$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin\theta}\frac{\partial^2\psi}{\partial\phi^2}\right]-\left[E-V(r)\right]\psi=0 \qquad (2)$$

Now the wavefunction ψ for a single electron in a central field can be written as the product of $R(r)$ – a radial factor which depends on $V(r)$ – and of $Y(\theta,\phi)$ which is an angular factor and, because of the spherical symmetry of the central force is independent of $V(r)$. $Y(\theta,\phi)$ depends on the magnitude and the z component of the angular momentum of the electron the former relating to l and the latter to m_l. Hence the spherical harmonic $Y(\theta\phi)$ is normally written as $Y_{lm_l}(\theta,\phi)$.

Applying the method of separation of variables where $\psi(r,\theta,\phi) = R(r)\,Y_{lm_l}$ and the separation constant is $l(l+1)$ we obtain from Equation (2):

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\cdot\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-\frac{l(l+1)}{r^2}\right]R-(E-V(r))=0 \qquad (3)$$

and

$$\frac{1}{\sin\theta}\cdot\frac{\partial}{\partial\theta}\cdot\sin\theta\frac{\partial}{\partial\theta}Y_{lm_l}+\frac{1}{\sin^2\theta}\frac{\partial^2 Y_{lm_l}}{\partial\phi^2}+l(l+1)Y_{lm_l}=0 \qquad (4)$$

The solution to Equation (4) has been considered in the lecture concerned with angular momentum and are the spherical harmonics

$$Y_{lm_l}(\theta,\phi) = P_{lm_l}(\theta)\,e^{im\phi} \qquad l=0,1,2\ldots$$

where $\qquad m_l = \pm l,\ldots,0$

$P_{lm_l}(\theta)$ are the associated Legendre functions. Values of $Y_{lm_l}(\theta,\phi)$ and their diagrams are given on the next page:.