RADIAL FUNCTION

Equation (3) contains expressions for the Coulomb potential $V(r)$ and the centrifugal potential $\frac{\ell(\ell+1)\hbar^2}{2mr}$

The former potential is attractive while the latter tends to keep the proton and electron apart

Equation (3) may in principle be solved by the use of a trial function of the form (polynomial) x (exponential). To do this the equation is first transformed by

(i) the substitution $R(r) = \frac{1}{r} u(r)$

(ii) using dimensionless variables eg $\rho = \frac{r}{a_0}$, $\beta^2 = E/E_1$ where a_0 is the Bohr radius and E_1 is the ground state energy of the hydrogen atom

Hence we obtain :-

$\rho \frac{d^2u}{d\rho^2} + 2u = -\rho\beta^2 u$ which with boundary conditions $u(\infty) = 0$ gives the solution $u = \rho^{\ell+1}\phi(\rho)e^{-\beta\rho}$ where $\phi(\rho)$ is an applied Laguerre polynomial. Solutions exist for $\beta = \frac{1}{n}$ where $n = n_r + \ell + 1$; energies are then defined as $E_n = E_1/n^2$ where n is the principal quantum number.

The explicit form of the radial function for the hydrogen atom is

$$R_{n,\ell} = -\left[\frac{2}{n}\frac{(n-\ell-1)!}{2n[(n+\ell)!]^3}\right]^{\frac{1}{2}}\left(\frac{2r}{n}\right)^{\ell} L_{n+\ell}^{2\ell+1}\left(\frac{2r}{n}\right)e^{-r/n}$$

Typical diagrams of the variation of the radial wave function with radius are given on the next page.

The wave functions of the hydrogen atom have the form $\psi(r,\theta,\phi) = R_{n\ell}(r)\,Y_{\ell m_\ell}(\theta,\phi)$