a) Yes. Here it is:

not required

$$(\hat{L}_x^2+\hat{L}_y^2)\hat{L}_z-\hat{L}_z(\hat{L}_x^2+\hat{L}_y^2)$$
$$=\hat{L}_x^2\hat{L}_z+\hat{L}_y^2\hat{L}_z-(\hat{L}_z\hat{L}_x^2)\hat{L}_x-(\hat{L}_z\hat{L}_y^2)\hat{L}_y$$
$$=\hat{L}_x^2\hat{L}_z+\hat{L}_y^2\hat{L}_z-(i\hbar\hat{L}_y+\hat{L}_x\hat{L}_z)\hat{L}_x-(-i\hbar\hat{L}_x+\hat{L}_y\hat{L}_z)\hat{L}_y$$
$$=\hat{L}_x^2\hat{L}_z+\hat{L}_y^2\hat{L}_z-i\hbar\hat{L}_y\hat{L}_x-\hat{L}_x\hat{L}_y-\hat{L}_x\hat{L}_z\hat{L}_x-\hat{L}_y\hat{L}_z\hat{L}_y$$
$$=\hat{L}_x^2\hat{L}_z+\hat{L}_y^2\hat{L}_z+(i\hbar)^2\hat{L}_z-\hat{L}_x(i\hbar\hat{L}_y+\hat{L}_x\hat{L}_z)-\hat{L}_y(\hat{L}_y\hat{L}_z-i\hbar\hat{L}_x)$$
$$=\hat{L}_x^2\hat{L}_z+\hat{L}_y^2\hat{L}_z-\hbar^2\hat{L}_z-i\hbar(\hat{L}_x\hat{L}_y-\hat{L}_y\hat{L}_x)-\hat{L}_x^2\hat{L}_z-\hat{L}_y^2\hat{L}_z$$
$$=-\hbar^2\hat{L}_z-i\hbar(i\hbar)\hat{L}_z$$
$$=-\hbar^2\hat{L}_z+\hbar^2\hat{L}_z=0 \Rightarrow((\hat{L}_x^2+\hat{L}_y^2)\hat{L}_z-\hat{L}_z(\hat{L}_x^2+\hat{L}_y^2))\psi=0$$

So $\hat{L}_x^2+\hat{L}_y^2$ commutes with $\hat{L}_z$

My working showed that $\hat{L}_z$ commutes with $\hat{L}_z^2$ and $\hat{L}_z^2$ will $\hat{L}_z^2$, but I'm not sure what to.

ditto.

b) No

yes!

$$(\hat{L}_x^2\hat{L}_z-\hat{L}_z\hat{L}_x^2)=\hat{L}_x^2\hat{L}_z-(\hat{L}_z\hat{L}_x)\hat{L}_x$$
$$=\hat{L}_x^2\hat{L}_z-(i\hbar\hat{L}_y+\hat{L}_x\hat{L}_z)\hat{L}_x$$
$$=\hat{L}_x^2\hat{L}_z-i\hbar\hat{L}_y\hat{L}_x-\hat{L}_x\hat{L}_z\hat{L}_x$$
$$=\hat{L}_x^2\hat{L}_z-i\hbar(\hat{L}_x\hat{L}_y-i\hbar\hat{L}_z)-\hat{L}_x(i\hbar\hat{L}_y+\hat{L}_x\hat{L}_z)$$
$$=\hat{L}_x^2\hat{L}_z-2i\hbar\hat{L}_x\hat{L}_y+(i\hbar)^2\hat{L}_z-\hat{L}_x^2\hat{L}_z$$
$$=-2i\hbar\hat{L}_x\hat{L}_y-\hbar^2\hat{L}_z \neq 0 \text{ in general}$$

$\therefore \hat{L}_x^2$ does not commute with $\hat{L}_z$

please give

Simplest way is to note that if $\hat{L}_x^2$ commutes with $\hat{L}_z$ then $\hat{L}_x$ must also commute with $\hat{L}_z$ & we know that will be true.

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