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4(d) $\langle L_z \rangle = \int_0^{2\pi}\int_0^{\infty} \psi^*(r,\phi)\,\hat{L}_z\,\psi(r,\phi)\,r\,dr\,d\phi$ ✓

There is no reason why R(r) should be real.

$= \int_0^{2\pi}\int_0^{\infty} \frac{R^*(r)}{2\sqrt{\pi}}(e^{-2i\phi}+e^{+2i\phi})\left(-i\hbar\frac{\partial}{\partial\phi}\right)\left(\frac{R(r)}{2\sqrt{\pi}}(e^{2i\phi}+e^{-2i\phi})\right)r\,dr\,d\phi$

changing order of integration $= \int_0^{\infty}\int_0^{2\pi} \frac{R^*(r)}{2\sqrt{\pi}}\times 2\cos 2\phi\left(-i\hbar\frac{\partial}{\partial\phi}\right)\left(\frac{R(r)}{2\sqrt{\pi}}2\cos 2\phi\right)r\,d\phi\,dr$

$= \int_0^{\infty}\int_0^{2\pi} \frac{R^*(r)}{2\sqrt{\pi}}\times 2\cos 2\phi \times -i\hbar\frac{R(r)}{2\sqrt{\pi}} \times -4\sin 2\phi \; r\,d\phi\,dr$

$= \int_0^{\infty}\int_0^{2\pi} \frac{-i\hbar}{\pi}|R|^2(r)\sin 4\phi \; r\,d\phi\,dr$

$= \int_0^{\infty}\left[\frac{i\hbar r R^2(r)}{4\pi}\cos 4\phi\right]_0^{2\pi} dr$

$= \int_0^{\infty}\left[\frac{i\hbar r R^2(r)}{4\pi}(\cos 8\pi - \cos 0)\right] dr$

$= 0.$ ✓

8/8 $\therefore \langle L_z \rangle = 0$

There is a much quicker way – see opposite – back of p. 11.

(e) We need to find $\langle L_z^2 \rangle$ – see opposite

$\langle L_z^2 \rangle = \int_0^{2\pi}\int_0^{\infty} \psi^*(r,\phi)\left((-i\hbar)^2\frac{\partial^2}{\partial\phi^2}\right)\psi(r,\phi)\,r\,dr\,d\phi$

$= \int_0^{2\pi}\int_0^{\infty} -\hbar^2\left(\frac{R^*(r)}{2\sqrt{\pi}}(e^{-2i\phi}+e^{2i\phi})\right)\frac{\partial^2}{\partial\phi^2}\left(\frac{R(r)}{2\sqrt{\pi}}(e^{2i\phi}+e^{-2i\phi})\right)r\,dr\,d\phi$

$= \int_0^{2\pi}\int_0^{\infty} -\hbar^2\left(\frac{R^*(r)}{2\sqrt{\pi}}\times 2\cos 2\phi\right)\frac{\partial^2}{\partial\phi^2}\left(\frac{R(r)}{2\sqrt{\pi}}\times 2\cos 2\phi\right)r\,dr\,d\phi$

$= \int_0^{2\pi}\int_0^{\infty} \frac{-\hbar^2|R|^2(r)}{4\pi}\times 2\cos 2\phi \times -8\cos 2\phi \; r\,dr\,d\phi$

$= \frac{+4\hbar^2}{\pi}\int_0^{2\pi}\int_0^{\infty}|R|^2(r)\,r\cos^2 2\phi\,dr\,d\phi.$

changing order of integration $= +\frac{4\hbar^2}{\pi}\int_0^{\infty}\int_0^{2\pi}|R|^2(r)\,r\cos^2 2\phi\,d\phi\,dr$