4a) The Schrödinger eqtn is

$$\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi = E\psi$$

In this case $V(x) = \frac{1}{2}Cx^2 = \frac{\hbar^2}{2ma^4}x^2$

$$\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{\hbar^2 x^2}{2ma^4}\psi = E\psi \quad \text{or} \quad -\frac{\partial^2\psi}{\partial x^2} + \frac{x^2\psi}{a^4} = \frac{2mE\psi}{\hbar^2} \quad (1)$$

If $\psi(x) = N\left(\frac{2x}{a}\right)\exp\left(\frac{-x^2}{2a^2}\right)$

$$\frac{\partial\psi}{\partial x} = N\frac{\partial}{\partial x}\left(\frac{2x}{a}\exp\left(\frac{-x^2}{2a^2}\right)\right)$$

$$= N\left(\frac{2}{a} - \frac{2x^2}{a^3}\right)\exp\left(\frac{-x^2}{2a^2}\right)$$

$$\frac{\partial^2\psi}{\partial x^2} = N\frac{\partial}{\partial x}\left(\left(\frac{2}{a} - \frac{2x^2}{a^3}\right)\exp\left(\frac{-x^2}{2a^2}\right)\right)$$

$$= N\left(\frac{-4x}{a^3} - \left(\frac{2}{a} - \frac{2x^2}{a^3}\right)\frac{x}{a^2}\right)\exp\left(\frac{-x^2}{2a^2}\right)$$

$$= N\left(\frac{-6x}{a^3} + \frac{2x^3}{a^5}\right)\exp\left(\frac{-x^2}{2a^2}\right)$$

Sub for $\frac{\partial^2\psi}{\partial x^2}$ and ψ into Schrödinger eqtn (1)

$$-\frac{\partial^2\psi}{\partial x^2} + \frac{x^2\psi}{a^4} = \frac{2mE}{\hbar^2}\psi$$

$$\frac{\partial^2\psi}{\partial x^2} = \left(\frac{x^2}{a^4} - \frac{2mE}{\hbar^2}\right)\psi$$

$$N\left(\frac{-6x}{a^3} + \frac{2x^3}{a^5}\right)\exp\left(\frac{-x^2}{2a^2}\right) = \left(\frac{x^2}{a^4} - \frac{2mE}{\hbar^2}\right)N\left(\frac{2x}{a}\right)\exp\left(\frac{-x^2}{2a^2}\right)$$

$$\frac{-6x}{a^3} + \frac{2x^3}{a^5} = \left(\frac{x^2}{a^4} - \frac{2mE}{\hbar^2}\right)\times\frac{2x}{a}$$

$$= \frac{2x^3}{a^5} - \frac{4mxE}{a\hbar^2}$$

So $\frac{-6x}{a^3} = \frac{-4mxE}{a\hbar^2}$

$$E = \frac{a\hbar^2}{4mx}\times\frac{6x}{a^3} = \frac{3\hbar^2}{2a^2m}$$

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b) For a classical oscillator the limits of displacement are given by the roots of $V(x) = E$

ie $\frac{\hbar^2}{2ma^4}x^2 = \frac{3\hbar^2}{2ma^2}$

$$x^2 = \frac{2ma^4}{\hbar^2}\times\frac{3\hbar^2}{2ma^2} = 3a^2$$