Solutions

Q1 $\boxed{\text{D,H}}$ An *acceptable* solution $\psi(x)$ is one for which $\psi(x)$ and its derivative $\psi'(x)$ are continuous and finite when $V(x)$ is finite and continuous (Section 6 of Unit 3). Option D is not necessarily true. $|\psi(x)|$ is an even function when $\psi(x)$ is even or odd, and this occurs necessarily only for bound state wavefunctions in symmetric wells, i.e when $V(x)$ is symmetric (even parity) about $x = 0$. Option H is true for bound solutions but not for unbound solutions.

Q2 $\boxed{\text{E}}$ The normalized ground state wavefunction $\psi_1(x)$ is given by FT Eq. 3-20 with $n = 1$. The probability density is $|\psi_1(x)|^2$ and so the required probability is

$$\begin{aligned}&\frac{2}{L}\int_{L/3}^{2L/3} \sin^2(\pi x/L)\,dx\\ &= \frac{2}{L}\int_{L/3}^{2L/3} \frac{1}{2}(1-\cos(2\pi x/L))\,dx\end{aligned}$$

where $\sin^2 A = (1-\cos 2A)/2$ has been used. Integrating gives

$$\begin{aligned}&\frac{1}{L}\left[x - \frac{L}{2\pi}\sin(2\pi x/L)\right]_{L/3}^{2L/3}\\ &= \frac{1}{3} + \sqrt{3}/2\pi.\end{aligned}$$

Q3 $\boxed{\text{G}}$ The normalized wavefunction for the first excited state is $\psi_2(x)$ from FT Eq. 3-20. The required probability is

$$\begin{aligned}&\frac{2}{L}\int_{L/6}^{5L/6} \sin^2(2\pi x/L)\,dx\\ &= \frac{2}{L}\int_{L/6}^{5L/6} \frac{1}{2}(1-\cos(4\pi x/L))\,dx\\ &= \frac{1}{L}\left[x - \frac{L}{4\pi}\sin(4\pi x/L)\right]_{L/6}^{5L/6}\\ &= \frac{2}{3} + \sqrt{3}/4\pi.\end{aligned}$$

Q4 $\boxed{\text{H}}$ The energy of the emitted photon is $E_4 - E_2$ where the E_n are the energies of the bound states given by FT Eq. 3-15. The frequency of the photon is obtained from the Planck–Einstein relation $E = h\nu$ and the wavelength is $\lambda = c/\nu$. Hence

$$\lambda = \frac{ch}{E_4 - E_2} = \frac{ch8mL^2}{h^2(4^2-2^2)} = \frac{\lambda_0}{12}.$$

Q5 $\boxed{\text{D,F}}$ Option D is false. Every measurement of the energy on any state will yield an energy eigenvalue. For the SHO the energy eigenvalues are odd multiples of the ground state energy. Option F is false. Stationary states of the SHO have even or odd parity. The probability density, $|\psi_n(x)|^2$, is therefore an even function for any stationary state and so the probability of finding the particle in $x \geqslant 0$ is always 1/2.

Q6 $\boxed{\text{F}}$ Look at Frame 9 on p. 11 of Unit 4 and you will see that the probability density $|\psi_n|^2$ has $(n+1)$ local maxima. $|\psi_6|^2$ has therefore 7 local maxima. The energy is $\left(n+\frac{1}{2}\right)\hbar\omega = \frac{13}{2}\hbar\omega$.

Q7 $\boxed{\text{C}}$ The wavefunction at any time $t > 0$ is given by
$\Psi(x,t) = 0.8\psi_1(x)e^{-3i\omega t/2} + 0.6\psi_2(x)e^{-5i\omega t/2}$
where the time-dependent factors are $e^{-iE_n t/\hbar}$ with the energy eigenvalues $E_n = (n+\frac{1}{2})\hbar\omega$ and $n = 1$ for the 1st excited state and $n = 2$ for the 2nd. (See the last equation of Sec. 3.3 on p. 8 of Unit 3 for the time dependence of stationary states, and Eq. 10 on p. 7 of Unit 5 for the time dependence of a superposition of two stationary states.) Evaluating $\Psi(x,t)$ at $t = \pi/\omega$ yields
$\Psi(x,\pi/\omega) = 0.8\psi_1(x)e^{-3i\pi/2} + 0.6\psi_2(x)e^{-5i\pi/2}$.
Now $e^{-3i\pi/2} = \cos(3\pi/2) - i\sin(3\pi/2) = i$ and $e^{-5i\pi/2} = -i$.
Hence option C is correct.

Q8 $\boxed{\text{C}}$ Referring to FT Table 4-1 it is seen that the x-dependent factor must be $e^{-x^2/2a^2}$ (i.e $n = 0$), and so the time-dependent factor must be $e^{-i\omega t/2}$. Hence $A = 1/a^2$ and $B = \omega$. The relationship between a and ω (denoted ω_0 in FT) is $a = (\hbar/m\omega)^{1/2}$. (See bottom of FT Table 4-1.) The ratio B/A is therefore $a^2\omega = \dfrac{\hbar}{m\omega}\omega = \hbar/m$, and so $m/\hbar = A/B$.

Q9 $\boxed{\text{C,E,H}}$ $\hat{p}_x$ does not commute with x. See bottom of p. 25 and Exercise 16 of Unit 5. $\hat{H}_0$ does not commute with x. To see this, work out $x\hat{H}_0 f$ and $\hat{H}_0 xf$ where f is some function.

$$\begin{aligned}x\hat{H}_0 f &= \frac{\hbar^2}{2m}x\frac{d^2f}{dx^2},\\ \hat{H}_0 xf &= \frac{\hbar^2}{2m}\frac{d^2}{dx^2}(xf)\\ &= \frac{\hbar^2}{2m}\frac{d}{dx}\left(f + x\frac{df}{dx}\right).\end{aligned}$$

Taking this a few steps further you can easily see that the two quantities are not equal and so $\hat{H}_0$ and x do not commute. $\hat{V}$ does commute with x because $x^2x = xx^2$. $\hat{H}$ does not commute with x because of the $\hat{H}_0$ term. $\hat{I}$ does commute with x because

$$(x\hat{p}_x - \hat{p}_x x)\,f = x\frac{\hbar}{i}\frac{df}{dx} - \frac{\hbar}{i}\frac{d}{dx}(xf) = i\hbar f$$

which is just a number times f. Note that we have just shown that $\hat{I} = i$. $\hat{J}$ does not commute with x because $\hat{J}$ is a constant times $\hat{p}_x$. Similarly for $\hat{K}$. Finally $\hat{L}$ does commute with x because $\hat{L} = \hat{I}^2 = i^2 = -1$, just a number.

Q10 $\boxed{\text{C,D}}$ $\hat{p}_x$ and $\hat{H}_0$ both commute with $\hat{p}_x$ since each is just a power of $\hat{p}_x$. $\hat{V}$ does not commute with $\hat{p}_x$ because x and $\hat{p}_x$ do not commute. (See answer to Q9). $\hat{H}$ does not commute with $\hat{p}_x$ because of the $\hat{V}$ term. $\hat{I}$ is just a number and therefore commutes with any operator. $\hat{J}$ is just a number $\hat{I}$ (see Q9 answer) times $\hat{p}_x$ and therefore commutes