*Q43 to Q45 mainly concern Unit 14.*

**Q43** This question concerns a perturbed hamiltonian $\hat{\mathbf{H}} = \hat{\mathbf{H}}^{(0)} + k\hat{\mathbf{H}}^{(1)}$ and two vectors

$$\phi = \phi^{(0)} + k\phi^{(1)} + k^2\phi^{(2)} + \cdots$$
$$\psi = \psi^{(0)} + k\psi^{(1)} + k^2\psi^{(2)} + \cdots$$

that satisfy $\hat{\mathbf{H}}\phi = \psi$. Given that $\hat{\mathbf{H}}^{(0)}\phi^{(0)} = \psi^{(0)}$, select from the key the *two* items that are correct equations. [Hint: Equate powers of $k$ in $\hat{\mathbf{H}}\phi = \psi$.]

KEY for Q43

A $(\phi^{(0)}, \hat{\mathbf{H}}^{(1)}\phi^{(0)}) = (\phi^{(0)}, \psi^{(1)}) + (\phi^{(0)}, \psi^{(1)})^*$

B $(\phi^{(0)}, \hat{\mathbf{H}}^{(1)}\phi^{(0)}) = (\phi^{(0)}, \psi^{(1)}) + (\phi^{(1)}, \psi^{(0)})^*$

C $(\phi^{(0)}, \hat{\mathbf{H}}^{(1)}\phi^{(0)}) = (\phi^{(0)}, \psi^{(1)}) - (\phi^{(0)}, \psi^{(1)})^*$

D $(\phi^{(0)}, \hat{\mathbf{H}}^{(1)}\phi^{(0)}) = (\phi^{(0)}, \psi^{(1)}) - (\phi^{(1)}, \psi^{(0)})^*$

E $(\phi^{(0)}, \hat{\mathbf{H}}^{(1)}\phi^{(1)}) = (\phi^{(0)}, \psi^{(2)}) + (\phi^{(0)}, \psi^{(2)})^*$

F $(\phi^{(0)}, \hat{\mathbf{H}}^{(1)}\phi^{(1)}) = (\phi^{(0)}, \psi^{(2)}) + (\phi^{(2)}, \psi^{(0)})^*$

G $(\phi^{(0)}, \hat{\mathbf{H}}^{(1)}\phi^{(1)}) = (\phi^{(0)}, \psi^{(2)}) - (\phi^{(0)}, \psi^{(2)})^*$

H $(\phi^{(0)}, \hat{\mathbf{H}}^{(1)}\phi^{(1)}) = (\phi^{(0)}, \psi^{(2)}) - (\phi^{(2)}, \psi^{(0)})^*$

Q44 and Q45 concern a hamiltonian $\hat{\mathbf{H}} = \hat{\mathbf{H}}^{(0)} + \delta\hat{\mathbf{H}}$, where the unperturbed hamiltonian, $\hat{\mathbf{H}}^{(0)}$, has a two-fold degenerate eigenvalue, $E$, corresponding to two orthonormal eigenvectors, $\psi_m^{(0)}$ and $\psi_n^{(0)}$, and the perturbation is described by an operator, $\delta\hat{\mathbf{H}}$, with matrix elements

$$(\psi_m^{(0)}, \delta\hat{\mathbf{H}}\psi_m^{(0)}) = (\psi_n^{(0)}, \delta\hat{\mathbf{H}}\psi_n^{(0)}) = -0.6A$$
$$(\psi_m^{(0)}, \delta\hat{\mathbf{H}}\psi_n^{(0)}) = (\psi_n^{(0)}, \delta\hat{\mathbf{H}}\psi_m^{(0)}) = +0.8A$$

where $A$ is a positive number.

**Q44** Select from the key the *two* items that form a pair of normalized vectors which solves the problem of degeneracy (in the sense of Section 4.2.5 of Unit 14).

KEY for Q44

| | | | |
|---|---|---|---|
| A | $\psi_m^{(0)}$ | E | $0.6\psi_m^{(0)} + 0.8\psi_n^{(0)}$ |
| B | $\psi_n^{(0)}$ | F | $0.6\psi_m^{(0)} - 0.8\psi_n^{(0)}$ |
| C | $(\psi_m^{(0)} + \psi_n^{(0)})/\sqrt{2}$ | G | $0.8\psi_m^{(0)} + 0.6\psi_n^{(0)}$ |
| D | $(\psi_m^{(0)} - \psi_n^{(0)})/\sqrt{2}$ | H | $0.8\psi_m^{(0)} - 0.6\psi_n^{(0)}$ |

**Q45** Select from the key the *two* items that are the first-order predictions of the energies of the states whose zero-order state vectors are given by your answers to Q44.

KEY for Q45

| | | | |
|---|---|---|---|
| A | $E + 0.2A$ | E | $E + 0.8A$ |
| B | $E - 0.2A$ | F | $E - 0.8A$ |
| C | $E + 0.6A$ | G | $E + 1.4A$ |
| D | $E - 0.6A$ | H | $E - 1.4A$ |

## PART E (Units 15 and 16)

*Q46 to Q55 mainly concern Unit 15.*

Q46 to Q50 share the same key and concern a rigid three-dimensional box, with sides of length $\frac{1}{2}L$, $L$ and $L$, that can be populated by two types of hypothetical particle: a boson, B, with spin $s = 0$ and mass $M$, and a fermion F, with spin $s = \frac{1}{2}$ and mass $M$. The ground-state energy of one B particle in the box is $E_0 = 3h^2/4ML^2$. The ground-state energy of one F particle in the box is also $E_0$. There are no interactions between particles. In each case, you are to select from the key the energy of the state in question. [Hint: the energy levels of a single B or F particle in the box are

$$E_{n,l,m} = \tfrac{1}{6}(4n^2 + l^2 + m^2)E_0$$

where $n, l, m$ are positive integers.]

**Q46** What is the ground-state energy of a system consisting of four B particles in the box?

**Q47** What is the ground-state energy of a system consisting of four F particles in the box?

**Q48** What is the ground-state energy of a system consisting of two B particles and two F particles in the box?

**Q49** What is the first excited-state energy of a system consisting of four B particles in the box?

**Q50** What is the first excited-state energy of a system consisting of four F particles in the box?

KEY for Q46 to Q50

| | | | |
|---|---|---|---|
| A | $4E_0$ | E | $6E_0$ |
| B | $\frac{9}{2}E_0$ | F | $\frac{13}{2}E_0$ |
| C | $5E_0$ | G | $7E_0$ |
| D | $\frac{11}{2}E_0$ | H | $\frac{15}{2}E_0$ |

**Q51** This question concerns the use of a variational method to estimate the energy of the lowest state of ortho-helium, using a trial wave function with spatial dependence

$$\phi(r_1, r_2) = (\psi_1(r_1)\,\psi_2(r_2) - \psi_1(r_2)\,\psi_2(r_1))/\sqrt{2}$$

where $\psi_1$ and $\psi_2$ are normalized wave functions of the 1s and 2s states respectively of the Coulomb hamiltonian with central charge $Z^*e$. A calculation gives

$$\langle E\rangle = \int\int \phi^*(r_1, r_2)\,\hat{\mathrm{H}}\,\phi(r_1, r_2)\,d\tau_1\,d\tau_2$$
$$= -E_{\mathrm{R}}Z^*\left(\tfrac{5}{2}Z - \tfrac{5}{4}Z^* - \tfrac{274}{729}\right)$$