Also

$$\hat{L}_z(x + iy)^2 = \frac{\hbar}{i}\frac{\partial}{\partial\phi}(r^2 e^{2i\phi})$$
$$= 2\hbar r^2 e^{2i\phi} \qquad \text{(eigenvalue} = 2\hbar)$$

and similarly for

$$(x - iy)^2 \qquad \text{(eigenvalue} = -2\hbar).$$

(Alternatively you could use the cartesian form of $\hat{L}_z = (\hbar/i)(x\partial/\partial y - y\partial/\partial x)$.)

Q21 [C,D] Try A: $\left(x\hat{L}_z - \hat{L}_z x\right)f$

$$= \frac{\hbar}{i}\left(r\cos\phi\frac{\partial}{\partial\phi} - \frac{\partial}{\partial\phi}r\cos\phi\right)f =$$

$-\frac{\hbar}{i}(r(-\sin\phi)f) \neq 0$ for any f. Therefore x does not commute with $\hat{L}_z$.
Try B: $(xy\hat{L}_z - \hat{L}_z xy)f =$
$$\frac{\hbar}{i}\left(r^2\sin\phi\cos\phi\frac{\partial}{\partial\phi} - \frac{\partial}{\partial\phi}r^2\sin\phi\cos\phi\right)f =$$
$-\frac{\hbar}{i}(r^2\cos 2\phi)f \neq 0.$

Try C: $\left(z\hat{L}_z - \hat{L}_z z\right)f = \frac{\hbar}{i}\left(z\frac{\partial}{\partial\phi} - \frac{\partial}{\partial\phi}z\right)f = 0$ so z
commutes with $\hat{L}_z$.
Similarly for D since $x^2 + y^2 + z^2 = r^2 + z^2$ which is not a function of ϕ.
The remaining options are all functions of ϕ, $g(\phi)$ say.
Then $\left(g\hat{L}_z - \hat{L}_z g\right)f = -\frac{\hbar}{i}\frac{\partial g}{\partial\phi}f \neq 0$ since $\frac{\partial g}{\partial\phi} \neq 0$
for any of options E–G.
($g = re^{i\phi}, re^{-i\phi}, r^2 e^{2i\phi}, r^2 e^{-2i\phi}$ for options E–G respectively). Note that as in Q20 you could have used $\hat{L}_z = (\hbar/i)(x\partial/\partial y - y\partial/\partial x)$ instead of $(\hbar/i)\partial/\partial\phi$.

Q22 [F,H] Spin states are represented by normalized spinors $\mathbf{a}$. (See Example 3 of Unit 9 and Postulate 1 on p. 12.) Try option A.

$$\mathbf{a}^\dagger\mathbf{a} = \frac{1}{4}[(1-i) \quad 1]\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$$
$$= \frac{1}{4}(2+1) = \frac{3}{4} \neq 1$$

and so the spinor is not normalized. Similarly none of options B, C, D, E and G are normalized. Now try option F.

$$\mathbf{a}^\dagger\mathbf{a} = \frac{1}{4}[1+i \quad 1-i]\begin{bmatrix} 1-i \\ 1+i \end{bmatrix}$$
$$= \frac{1}{4}(2+2) = 1$$

and similarly for option H.

Q23 [B,E] Multiply option B by $-i$,
$-i\frac{1}{2}[(1+i) \quad i]^T = \frac{1}{2}[(-i+1) \quad 1]$ which is option E.

Q24 [D] Spinors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if $\mathbf{a}^\dagger\mathbf{b} = 0$. For option D we have

$$\frac{1}{2}[(1-i) \quad 2]\begin{bmatrix} 1+i \\ -1 \end{bmatrix} =$$
$$\frac{1}{2}[(1-i)(1+i) + 2 \times (-1)] = 0$$

Q25 [H] The given state is $\mathbf{e}_+(\mathbf{n})$ given by (see Eq. 27a of Unit 9)

$$\mathbf{e}_+(\mathbf{n}) = \left[\cos(\theta/2) \quad \sin(\theta/2)e^{i\phi}\right]^T$$
$$= \left[\cos(\pi/6) \quad \sin(\pi/6)e^{i\pi/4}\right]^T$$
$$= \left[\sqrt{3}/2 \quad (1/2)\frac{1}{\sqrt{2}}(1+i)\right]^T$$

The required probability is $p = \left|\mathbf{e}_+^\dagger(\mathbf{n})\mathbf{e}(\mathbf{j})\right|^2$ where $\mathbf{e}(\mathbf{j}) = [1/\sqrt{2} \quad i/\sqrt{2}]^T$ (from Postulate 4 and using Eq. 27a with $\theta = \phi = \pi/2$ to give $\mathbf{n} = $ unit vector $\mathbf{j}$ along the y-axis). Hence

$$p = \left| [\sqrt{3}/2 \quad (1/2)(1/\sqrt{2})(1-i)]\begin{bmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix} \right|^2$$
$$= \left| \frac{\sqrt{3}}{2\sqrt{2}} + \frac{i(1-i)}{4} \right|^2$$
$$= \frac{1}{2} + \frac{1}{4}\left(\frac{3}{2}\right)^{1/2}$$
$$= 0.806$$

Q26 [F] $\mathbf{a}^\dagger\mathbf{b} = [i \quad 0 \quad 1]\begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix} = i \times (-i) +$

$(0 \times 1) + (1 \times 0) = 1.$
(See Section 1.3 of Unit 10, Exercise 5 for example.)

Q27 [E] $\mathbf{b}^\dagger\mathbf{Ma} = [i \quad 1 \quad 0]\begin{bmatrix} 0 & 1 & i \\ 0 & i & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} -i \\ 0 \\ 1 \end{bmatrix}$

$$= [i \quad 1 \quad 0]\begin{bmatrix} 0 \times (-i) + 1 \times 0 + i \times 1 \\ 0 \times (-i) + i \times 0 + 1 \times 1 \\ 1 \times (-i) + 0 \times 0 + 1 \times 1 \end{bmatrix}$$

$$= [i \quad 1 \quad 0]\begin{bmatrix} i \\ 1 \\ -i+1 \end{bmatrix} = -1 + 1 + 0$$

$= 0.$ (See Section 1.3 of Unit 10.)

Q28 [E] $\mathbf{b}^\dagger\mathbf{M}^\dagger\mathbf{a} = [i \quad 1 \quad 0]\begin{bmatrix} 0 & 0 & 1 \\ 1 & -i & 0 \\ -i & 1 & 1 \end{bmatrix}\begin{bmatrix} -i \\ 0 \\ 1 \end{bmatrix}$

$$= [i \quad 1 \quad 0]\begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix} = i - i = 0.$$

Q29 [B] $(i\mathbf{b})^\dagger\mathbf{a} = -i[i \quad 1 \quad 0]\begin{bmatrix} -i \\ 0 \\ 1 \end{bmatrix} =$

$-i(i \times (-i) + 1 \times 0 + 0 \times 1) = -i.$
(See Section 1.3 of Unit 10.)