

## Question 4 - 25 marks

Block II, Chapter 4.

A metal plate of thickness  $L$  is initially heated to a uniform temperature  $T = T_0$  and is then plunged (at time  $t = 0$ ) into a bath of water and ice, at temperature  $T = 0$ . In the following you can assume that water is an excellent conductor of heat so that the temperature of the surfaces of the plate in contact with the water (at  $z = 0$  and  $z = L$ ) is  $T = 0$  for all times when the plate is in the water. The thermal diffusivity of the plate is  $D$ .

- (a) Use separation of variables to find solutions of the diffusion equation of the form  $T(z,t) = f(z)g(t)$  that satisfy the appropriate boundary conditions.

(b) A general solution of the diffusion equation is a linear combination of the solutions obtained in part (a) in the form

$$T(z,t) = \sum_{n=1}^{\infty} a_n f_n(z) \exp(-\lambda_n t).$$

Give expressions for the decay coefficients  $\lambda_n$  and for the normalised eigenfunctions  $f_n(z)$ .

- (c) Determine the coefficient  $a_1$ . (Hint: Use the approach described in Subsection 4.3.5.)

(d) In the long-time limit, the temperature is well-approximated by an expression that involves only one term of the sum. Write down this approximation for the temperature in the long-time limit.

(e) The plate is withdrawn from the water bath at time  $t_f$ , which is sufficiently large that the approximation considered in part (d) is applicable. If the loss of heat to the air can be neglected, the temperature of the plate will reach a uniform value  $T_f$  throughout its thickness. Explain how the value of  $T_f$  may be determined, and show that

$$T_f = \frac{\pi^2}{8} T_0 \exp(-\pi^2 D t_f / L^2).$$

[4]

$$B_n \int_0^L \sin^2 n \pi z \, dz = \frac{L}{2} B_n \left[ \frac{1}{2} (1 - \cos 2n\pi z) \right]_0^L$$

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