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Question 1 – 25 marks

Block II, Chapter 1.

(a) A conference organiser is considering hiring a vending machine for use at an event that will last for 50 hours. Hiring a machine costs £750, and she estimates that the machine will generate £25 of income for each hour of use. If the machine is perfectly reliable, is it worth hiring one? [6]

(b) The probability density for this machine to fail after t hours of use is $\rho(t) = \exp(-t/\tau)/\tau$, where τ is a positive constant. Show that the probability that a failure will have occurred after T hours is $P(T) = 1 - \exp(-T/\tau)$. [6]

(c) If the machine is not totally reliable, the organiser must consider the expectation value of the time for which it will be in operation.

The machine is not required to operate for more than $T_{\max}$ hours (where, in this example, $T_{\max} = 50$ hours). The mean time that the machine is in operation is obtained by combining two contributions:

- The machine works up to time $T_{\max}$ with probability $1 - P(T_{\max})$, which gives a contribution $T_{\max}[1 - P(T_{\max})]$.
- The case where the machine fails before $T_{\max}$ gives a contribution that is an integral over the time to failure, t.

Then the expectation value of the time that the vending machine is used is

$$\langle t \rangle = \int_0^{T_{\max}} dt\, t\, \rho(t) + [1 - P(T_{\max})]T_{\max}.$$

Perform the integration and simplify the result to obtain an expression for $\langle t \rangle$. [8]

(d) If $\tau = 40$ hours and the machine cannot be repaired before the end of the event, what is the expected profit from hiring the vending machine? [5]

Question 2 – 25 marks

Block II, Chapter 2.

Consider a random process generating a sequence of numbers x_n:

$$x_{n+1} = \alpha x_n + r_n,$$

where $0 < \alpha < 1$ is a constant and the r_n are independent random numbers with a uniform probability distribution on the interval $[0, 1]$. The initial condition is $x_0 = 0$.

(a) Calculate the mean $\langle r \rangle$, second moment $\langle r^2 \rangle$ and variance of the random numbers r_n. [4]

(b) Show that the mean and second moment of x_n satisfy

$$\langle x_{n+1} \rangle = \alpha \langle x_n \rangle + \langle r \rangle,$$
$$\langle x_{n+1}^2 \rangle = \alpha^2 \langle x_n^2 \rangle + 2\alpha \langle x_n \rangle \langle r \rangle + \langle r^2 \rangle.$$

(*Hint*: Note that the statistics of the r_n are the same for each value of n, so we can replace $\langle r_n \rangle$ by $\langle r \rangle$. Also, consider whether the random variables x_n and r_n are independent.) [6]