Square root of the Lie derivative

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Abstract

In this short note we show that the Lie derivative acting on differential forms can be written as the "sqaure" of the sum of the exterior derivative and the interior product.

1 Differential forms as superfunctions

We will *define* differential forms on a manifold M to be functions on the supermanifold ΠTM. This needs a little explanation. For our purposes here, a supermanifold is a "manifold" with extra anticommuting coordinates. This will become clearer as we proceed.

The supermanifold ΠTM is known as the *antitangent bundle.* The functor Π is the parity reverse functor. It shifts the parity of the fibre coordinates of a vector bundle. Consider the tangent bundle, which we equip with natural local coordinates $\{x^A, v^A\}$. Here under a change of coordinates on the base we have the transformation law

$$\begin{aligned} x^A &\rightarrow \overline{x}^A(x); \\ v^A &\rightarrow \overline{v}^A = v^B\left(\frac{\partial \overline{x}^A}{\partial x^B}\right). \end{aligned} \tag{1}$$

The important thing here is that these coordinates *commute*; $x^Ax^B = x^Bx^A$ and $v^Av^B = v^Bv^A$.

The anticotangent bundle comes with natural local coordinates $\{x^A, dx^A\}$. Now the fibre coordinates are odd, that is they anticommute; $dx^Adx^B = -dx^Bdx^A$. The transformation rules are exactly the same as for the tangent bundle.

We will define **differential forms** on M to be functions on the total space of ΠTM. Then a differential form (locally) is given by the formal Maclaurin series

$$\begin{aligned} \omega(x, dx) &= \omega_0(x) + dx^A\omega_A(x) + \frac{1}{2!}dx^Adx^B\omega_{BA}(x) \\ &+ \frac{1}{3!}dx^Adx^Bdx^C\omega_{CBA}(x) + \cdots . \end{aligned} \tag{2}$$

We will use the following notation to define the space of (pseudo)forms as

$$\Omega^*(M) = C^\infty(\Pi TM). \tag{3}$$

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The **exterior derivative** is defined as the homological vector field given in local coordinates by

$$d = dx^A \frac{\partial}{\partial x^A}. \tag{4}$$

Note that with this definition we lose the usual standard grading of differential forms. That is, we do not insist on differential forms to be polynomial in dx. However, a filtration can be introduced on the space of differential forms that are polynomial in fibre coordinates which we will refer to as **form degree**, simply as the polynomial degree in dx. A differential form shall be called a **p-form** if it is a monomial of degree p in dx.

The **interior product** is given by the homological vector field

$$i_X = X^A \frac{\partial}{\partial dx^X}, \tag{5}$$

where $X = X^A \frac{\partial}{\partial x^A} \in Vect(M)$ is the vector field in question. Note that the derivatives with respect to dx inherit the the "oddness". This must be taken into account when acting on differential forms. For example, $\frac{\partial}{\partial dx^A} dx^B dx^C = \delta_A^B dx^C - dx^B \delta_A^C$.

The **Lie derivative** can then be defined using Cartan's homotopy formula,

$$L_X = [d, i_X], \tag{6}$$

the bracket here is the graded commutator. As both the exterior derivative and the interior product are odd operators, the above is $d \circ i_X + i_X \circ d$. It is left to an exercise for the reader to work out the local expression of the Lie derivative as a vector field.

Remark As far as the author is aware writing Cartan calculus in terms of functions on a supermanifold first appeared in [1]. However, in Cartan's original work it does look like he was considering derivatives with respect to odd variables!

2 Square root of the Lie derivative

Proposition 2.1 *The operator $d_X = d + i_X \in End(\Omega^*(M))$ squares to the Lie derivative.*

Proof This can be done via direct computation

$$\begin{aligned}
(d_X)^2 = \frac{1}{2}[d_X, d_x] &= (d + i_X)(d + i_X) \\
&= d \circ i_X + i_X \circ d + d^2 + i_X^2 \\
&= L_X,
\end{aligned} \tag{7}$$

where we have used Cartan's formula and the fact that the exterior derivative and the interior product square to zero.

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Remark This generalises to M being a supermanifold, but only if the vector field in question is even.

3 Remarks

The square root of the Lie derivative appears to be know by those who know. That is the author has not seen it in print. One is reminded of the Dirac equation and maybe more importantly (super)connections and their curvature.

References

[1] Th. Th. Voronov. Geometric integration theory on supermanifolds. Sov. Sci. Rev. C. Math. Phys. **9**. Harwood Academic Publ., (1992) 142p.