M2S2 Spring 12 - Assessed Coursework Sheet 1, Spring 2012

To be handed in no later than Friday, 10th Februrary 2012, 2 pm.
Please hand in to the Mathematics General Office

1. **[9 points]** Suppose we observe a realisation $y_1, \ldots, y_n$ of the iid random variables $Y_1, \ldots, Y_n$ which follow a uniform distribution on $[\theta, \theta + 1]$. The unknown parameter is $\theta \in \mathbb{R}$.

   (a) Carefully write down the pdf f_θ of Y_1.

   (b) Write down the likelihood function L of the observation, including its domain. Simplify the likelihood function as far as possible.

   (c) Find the maximum likelihood estimator(s).

   (d) Suppose $n = 4$ and suppose we observe the realisation $0.5, 0.3, 1.1, 1.2$. Give all maximum likelihood estimates of θ.

   (e) Suppose we observed the realisations $0.5, 0.1, 1.1, 1.2$. What can be said about the model for this realisation?

   (f) Consider $T = \max(Y_1, \ldots, Y_n) - \theta$. Is T a pivotal quantity for θ?

   (g) Construct a 95% confidence interval for θ.

2. **[5 points]** Suppose from an infinite population of students we pick independently $n = 100$ students and ask them a multiple choice question with two possible answers, one of which is correct. We observe the number y of correct answers. If a students knows the correct answer then she/he answers correctly. If a student does not know the correct answer then she/he picks one of the answers with equal probability.

   Let $\theta \in [0, 1]$ be the probability that a students knows the correct answer. Let ϕ be the probability that a student answers correctly.

   (a) How are ϕ and θ related? What values can ϕ take?

   (b) Derive the maximum likelihood estimator $\hat{\phi}$ for ϕ.

   (c) Derive the maximum likelihood estimator $\hat{\theta}$ for θ.

   (d) Construct a 95% asymptotic confidence interval for θ. What is this confidence interval for $y = 59$, $y = 60$ and $y = 70$?

   You may use results from the lecture in this question.

3. **[6 points]** Suppose $n = 100$ students answer a multiple choice question with two possible answers. We observe the number y of correct answers. Suppose if a students knows the correct answer then she/he answers correctly. If a student does not know the correct answer then she/he picks one of the answers with equal probability.

   Let $\theta \in \{0, \ldots, n\}$ be the number of students that know the correct answer.

   (a) Write down the likelihood function. Make sure you define it for all possible θ.

   (b) Suppose we observe 70 correct answers. Plot the likelihood function and determine the maximum likelihood estimate.

   (c) Consider the hypothesis $H_0 : \theta = 0$ against $H_1 : \theta > 0$.
   What is the distribution of Y, the number of correct answers, under the null hypothesis? Construct a test to the level 5%. Clearly state the rejection region.

   You may use any programming language of your choice to answer this question.