MST326

Assignment Booklet I 2012J

Contents **Cut-off date**

Please send all your answers to each tutor-marked assignment (TMA), together with an appropriately completed assignment form (PT3), to reach your tutor on or before the appropriate cut-off date shown above.

You will find instructions on how to fill in the PT3 form in the current *Assessment Handbook*. Remember to fill in the correct assignment number as listed above, and allow sufficient time in the post for each assignment to reach its destination on or before the cut-off date.

The marks allocated to each part of a TMA question are indicated in brackets in the margin.

Please show your working for all questions. This will give your tutor the opportunity to award you some marks for a question where your working is partially correct even though you may not have the correct final answer.

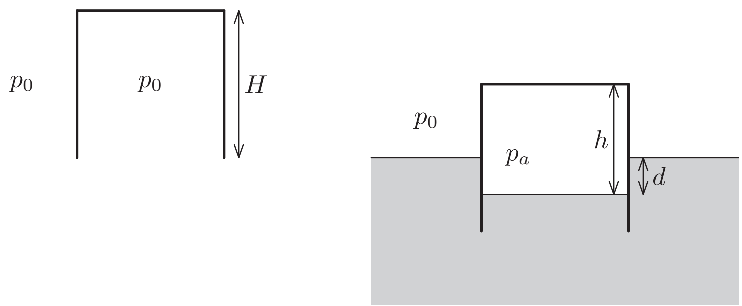
You are advised to keep a copy of your assignments in case of loss in the mail.

This assignment covers Block 1 (*Units 1* to *4*). Each question is allotted 25 marks.

Question 1 (*Unit 1*)

(a) An inverted cylindrical container, closed at the top and open at the bottom, has height H and cross-sectional area A. Initially, the cylinder contains air at atmospheric pressure, p_0 (see the left-hand part of the vertical cross-sectional diagram below).

The container is now lowered slowly into a large tank of water, until an equilibrium position is reached at which it floats without further motion. (This will be a stable configuration provided that the ratio $\sqrt{A} : H$ is sufficiently large.) The water level within the container is now a distance h below the top of the container and at a depth d below the water level outside the container (see the right-hand part of the diagram below). The thickness of the container is negligible. The water has constant density ρ, and the magnitude of the acceleration due to gravity is g.



(i) Write down an expression for the water pressure at a depth d below the water level outside the container. [*1*]

(ii) By applying Boyle's Law, express the air pressure p_a within the partly submerged container in terms of p_0, h and H. [*3*]

(iii) By considering the continuity of pressure across the air–water interface within the container, deduce that

$$h = \frac{H}{1 + \alpha d}, \qquad \text{where } \alpha = \frac{\rho g}{p_0}.$$ [*2*]

(iv) By considering all of the external vertical forces acting on the container and the air within it, express the mass M of the container in terms of ρ, d and A. Explain how your answer is equivalent to a statement of Archimedes' Principle. [*4*]

(v) Show that the container will float (as described, with the top of the container above the outside water level) provided that

$$M < \tfrac{1}{2}\rho A \alpha^{-1}\left(-1 + \sqrt{1 + 4\alpha H}\right).$$ [*4*]

(vi) Hence estimate the maximum possible mass of a floating container with height 2 m and cross-sectional area $50\,\mathrm{m}^2$, taking $p_0 = 10^5\,\mathrm{Pa}$, $\rho = 10^3\,\mathrm{kg\,m^{-3}}$ and $g = 10\,\mathrm{m\,s^{-2}}$. [*1*]

Comment related to part (a): This type of calculation could be used to estimate the maximum load to be carried by a hovercraft, to ensure that it floats when its engines are switched off.

(b) The Earth's atmosphere is composed of several layers. The *mesosphere*, which is regarded as extending from 51 km to 85 km above the Earth's surface, is characterised by a steadily decreasing temperature as height increases. Accepted values for the air temperature are 271 K at the base of the mesosphere and 187 K at the top, and the pressure at the base is taken to be 67 Pa.

A model of the mesosphere assumes that:

- the air is static;
- the perfect gas law applies, with gas constant $R = 287$ in SI units;
- the magnitude g of the acceleration due to the gravity is constant, with $g = 9.81\,\mathrm{m\,s^{-2}}$;
- if z (in metres) is the height above the surface of the Earth, then the temperature distribution $\Theta(z)$ (in K) is a linear function between the two temperature values given above, within the height range $5.1 \times 10^4 \le z \le 8.5 \times 10^4$.

(i) Find the appropriate linear function $\Theta(z) = a - bz$, where a and b are positive constants to be evaluated. (Here and below, give inexact numerical values to four significant figures, while using full calculator accuracy to perform all calculations.) *[3]*

(ii) Find the corresponding pressure distribution $p(z)$ within the mesosphere. (It is easiest to assign values to the various parameters only at the final stage of the argument.) *[5]*

(iii) Hence show that, according to this model, the atmospheric pressure at the top of the mesosphere is less than 1% of its value at the base. *[2]*

Comment related to part (b): The mesosphere lies above the troposphere (up to roughly 20 km, where the temperature decreases with height) and the stratosphere (from 20 km up to roughly 50 km, where the temperature increases with height). The top of the mesosphere is the coldest region of the entire atmosphere of the Earth.

Question 2 (*Unit 2*)

(a) (i) Find the general solution of the Cauchy–Euler equation

$$x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}-3y=0 \qquad (x>0).$$ [3]

(ii) Show that the Wronskian of two independent solutions of this Cauchy–Euler equation is a multiple of $1/x^3$. [1]

(iii) Use the method of variation of parameters to find the general solution of the differential equation

$$x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}-3y=8x \qquad (x>0).$$ [5]

(b) The eigenvalue problem

$$x^2\,u''(x)-3x\,u'(x)+(4+\lambda)\,u(x)=0, \qquad u(1)=0,\ u(e^2)=0,$$

has only positive eigenvalues λ, and the general solution of the differential equation for $\lambda=\omega^2$ (with $\omega>0$) is

$$u(x)=x^2[A\cos(\omega\ln x)+B\sin(\omega\ln x)],$$

where A and B are arbitrary constants. (You are *not* asked to show this.)

Find all the eigenvalues and eigenfunctions of the problem. [5]

(c) This part concerns the differential equation

$$(1-x^2)y''+xy'+3y=0 \qquad (-1<x<1).$$

(i) Show that if $y=\sum_{j=0}^{\infty}a_jx^j$ is the general solution in power series form, then

$$a_{j+2}=\frac{j-3}{j+2}\,a_j \qquad (j=0,1,2,\ldots).$$ [4]

(ii) By choosing $a_0=0$ and $a_1=1$, deduce that the differential equation has a polynomial solution, $y_1(x)$. [2]

(iii) Use the method of reduction of order, together with the solution $y_1(x)$ found in part (c)(ii), to find the general solution of the differential equation.

Hint: You may use without proof the result that

$$\int\frac{\sqrt{1-x^2}}{x^2(1-\frac{2}{3}x^2)^2}\,dx=-\frac{(1-x^2)^{3/2}}{x(1-\frac{2}{3}x^2)}+c,$$

where c is an arbitrary constant. [5]

Question 3 (*Unit 3*)

Note that the answers to each of parts (b) and (c) of this question can be checked by substituting the solution found into the given equations. (No marks will be awarded for doing this in part (a).)

(a) Show that the partial differential equation

$$\frac{\partial u}{\partial x}+\frac{1}{3x}\,u=\tfrac{20}{3}x^2 \qquad (x>0),$$

for $u(x,y)$, has the general solution

$$u(x,y)=2x^3+x^{-1/3}f(y),$$

where f is an arbitrary function. [5]

(b) (i) Use the method of characteristics, together with your answer to part (a), to find the general solution for $u(x, y)$ of the partial differential equation

$$3x\,\frac{\partial u}{\partial x} + 6xy\,\frac{\partial u}{\partial y} + u = 20x^3 \qquad (x > 0,\ y > 0). \qquad [12]$$

(ii) Find also the particular solution of this equation that satisfies the condition

$$u = y + 2 \qquad \text{on } x = 1. \qquad [3]$$

(c) Using your answer to part (a), find the general solution for $u(x, y)$ of each of the following partial differential equations.

(i) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{1}{3x}\,\dfrac{\partial u}{\partial x} = \tfrac{20}{3}x^2 \qquad (x > 0,\ y > 0)$ [2]

(ii) $\dfrac{\partial^2 u}{\partial x\,\partial y} + \dfrac{1}{3x}\,\dfrac{\partial u}{\partial y} = \tfrac{20}{3}x^2 \qquad (x > 0,\ y > 0)$ [2]

(iii) $\dfrac{\partial^2 u}{\partial x\,\partial y} + \dfrac{1}{3y}\,\dfrac{\partial u}{\partial x} = \tfrac{20}{3}y^2 \qquad (x > 0,\ y > 0)$ [1]

Question 4 (*Unit 4*)

(a) (i) By evaluating $\mathbf{curl}\,\mathbf{F}$, show that the vector field given in cylindrical polar coordinates by

$$\mathbf{F}(r, \theta, z) = 3r^2 \sin\theta \cos(2\theta)\,\mathbf{e}_r + r^2 \cos\theta\,(3\cos(2\theta) - 2)\,\mathbf{e}_\theta + 2z\,\mathbf{e}_z$$

is irrotational. [3]

(ii) Determine a scalar potential $\phi(r, \theta, z)$ for this vector field $\mathbf{F}$. [6]

(iii) Evaluate the line integral $\int_{AB} \mathbf{F} \cdot d\mathbf{r}$, where AB is the line segment from the point A $(0, 3, 0)$ to the point B $(-1, 0, 2)$, given in Cartesian coordinates. [3]

(b) For the vector field $\mathbf{u}$ given in Cartesian coordinates by

$$\mathbf{u} = x^2 y\,\mathbf{i} - 3x\,\mathbf{j},$$

find $(\mathbf{u} \cdot \boldsymbol{\nabla})\,\mathbf{u}$.

Hint: See the margin of page 198 in *Unit 4* for the definition of $\mathbf{a} \cdot \boldsymbol{\nabla}$. [2]

(c) Verify Gauss' Theorem,

$$\int_B \operatorname{div} \mathbf{F}\,dV = \oint_S \mathbf{F} \cdot \mathbf{n}\,dA,$$

for the vector field

$$\mathbf{F}(r, \theta, \phi) = r^2(\cos\theta\,\mathbf{e}_r - 3\sin\theta\,\mathbf{e}_\theta + 2\,\mathbf{e}_\phi)$$

(in spherical polar coordinates), where the region B is the closed hemisphere given by

$$0 \le r \le 1, \qquad \tfrac{1}{2}\pi \le \theta \le \pi, \qquad -\pi < \phi \le \pi,$$

and S is the surface that encloses B.

Hints: On the flat (top) portion of S, the variables r, ϕ act as plane polar coordinates, with area element $\delta A = r\,\delta r\,\delta\phi$. Note also that the outward unit normal vector here points upwards. [11]

This assignment covers Block 2 (*Units 5* to *8*). Each question is allotted 25 marks.

Question 1 (*Unit 5*)

Consider the two-dimensional vector field

$$\mathbf{u} = 6y(9yt - x)\,\mathbf{i} + 3y^2\,\mathbf{j}.$$

(a) Show that $\mathbf{u}$ could represent the velocity vector field of an incompressible fluid flow. [2]

(b) (i) By solving appropriate differential equations, show that the pathlines of this flow for $y \neq 0$ are described by the equations

$$x = B(3t + A)^2 - 3 + \frac{2A}{3t + A}, \qquad y = -\frac{1}{3t + A},$$

where A and B are arbitrary constants.

Hint: Note that

$$\frac{3t}{(3t + A)^4} = \frac{1}{(3t + A)^3} - \frac{A}{(3t + A)^4}.$$ [6]

(ii) For the pathline that passes through the point $(3, 1)$ at time $t = 0$, eliminate t between the equations given in part (b)(i) to obtain an explicit equation of the form $x = x(y)$ for the pathline. [3]

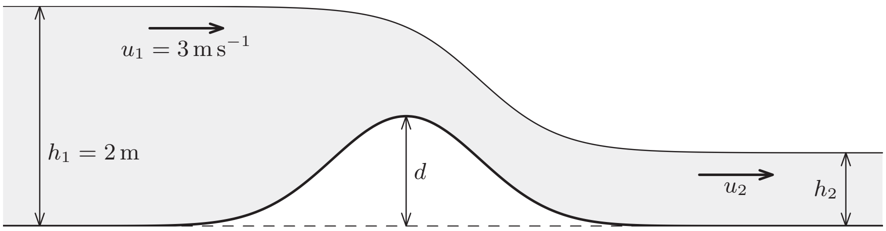
(c) (i) Write down the equations describing the stream function for the velocity vector field $\mathbf{u}$. Hence find the stream function for this flow, and the equations of the streamlines. [7]

(ii) Find in particular the equation of the streamline at time $t = 0$ that passes through the point $(3, 1)$, in the form $x = x(y)$. [2]

(iii) Sketch on a single diagram the pathline whose equation was found in part (b)(ii) and the streamline whose equation was found in part (c)(ii). Indicate in each case the asymptotic behaviour of the curve and the direction of flow along it. [5]

Question 2 (*Unit 6*)

Water flows steadily along a horizontal open channel of uniform width, over a broad-crested weir that rises to a height d metres above the upstream and downstream level of the channel floor, as shown in the figure below. Upstream from the weir (to the left), the water depth is $h_1 = 2$ metres and the speed is $u_1 = 3\,\mathrm{m\,s^{-1}}$. Take the magnitude of the acceleration due to gravity as $g = 10\,\mathrm{m\,s^{-2}}$.



(a) Calculate the Froude number for the flow upstream, and hence show that the flow there is subcritical. [*3*]

(b) Find the volume flow rate per unit width, Q (in $\mathrm{m^2\,s^{-1}}$), and the specific energy E (in m) for the flow upstream. [*3*]

(c) Show that the depth h_2 in the downstream section of the channel satisfies the equation

$$20h_2^3 - 49h_2^2 + 36 = 0.$$

Supposing that the flow downstream from the weir is supercritical (as illustrated above), find the depth h_2 and speed u_2 of the flow downstream. [*6*]

In parts (d) and (e), give numerical answers to two decimal places.

(d) Find the critical depth, h_c, for the specific energy function that corresponds to the volume flow rate per unit width Q found in part (b). State the minimum possible value, $E_{\min}$, for the specific energy. [*4*]

(e) Hence, by applying Bernoulli's equation along a suitable streamline, determine the height d metres of the weir that ensures supercritical flow downstream. [*4*]

(f) Suppose now that the height of the weir is reduced to 0.1 m. In steady flow, with the same values of h_1 and u_1 as before, the flow will be subcritical downstream, as illustrated in Figure 3.18 on page 97 of *Unit 6*.

Find the height by which the water surface dips when above the crest of the weir, as compared to the water surface level upstream and downstream. Give your answer to three decimal places. [*5*]

Question 3 (*Unit 7*)

A flow is represented, in cylindrical polar coordinates (r, θ, z), by the stream function

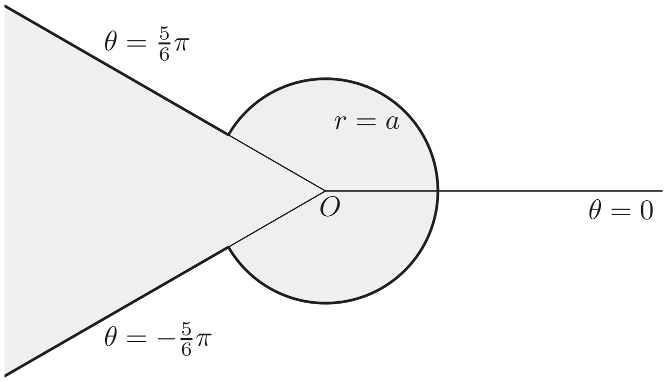
$$\psi = \tfrac{5}{3}U\left(r - \frac{a^2}{r}\right)\cos\left(\tfrac{3}{5}\theta\right),$$

where U is a positive constant.

(a) Find the velocity components of the flow, u_r and u_θ. [2]

(b) Show that this stream function could model a two-dimensional flow of an inviscid, incompressible fluid in the region to the right of a boundary (shown in cross-section below) consisting of:

- five sixths of a cylinder of radius a, whose axis lies along the z-axis;
- flat surfaces perpendicular to the (r, θ)-plane, along $\theta = \pm\frac{5}{6}\pi$, extending indefinitely outwards from the cylindrical surface. [3]



(c) By considering the values of u_r and u_θ for large r at $\theta = \pm\frac{5}{6}\pi$, and their values for any $r > a$ at $\theta = 0$, or otherwise, sketch a few typical streamlines of the flow, and indicate the direction of flow along them. [4]

(d) Assume that the body force is zero on the boundary, in which case it can be shown (you are not asked to do so) that Bernoulli's equation in the form $p/\rho + \frac{1}{2}u^2 = \text{constant}$ is valid along the streamlines formed by the boundary. Use this fact to show that the net surface force per unit length on the five-sixths cylinder, due to the flow, is

$$a\left(\tfrac{100}{11}\rho U^2 - p_0\right)\mathbf{i},$$

where ρ is the density of the fluid and p_0 is the stagnation pressure.

Hint: You may use without proof the result that

$$\int \cos^2\left(\tfrac{3}{5}\theta\right)\cos\theta\, d\theta = \tfrac{5}{4}\sin\left(\tfrac{1}{5}\theta\right) + \tfrac{1}{2}\sin\theta + \tfrac{5}{44}\sin\left(\tfrac{11}{5}\theta\right).$$ [9]

(e) Show that the vorticity of the flow is

$$\frac{16U}{15r}\left(\frac{a^2}{r^2} - 1\right)\cos\left(\tfrac{3}{5}\theta\right)\mathbf{e}_z.$$ [2]

(f) The plane surface S is the region $0 \le \theta \le \frac{5}{9}\pi,\ a \le r \le 4a$ (a sector of an annulus), and the closed curve C is the boundary of S. Using the result of part (e), or otherwise, show that the circulation of the velocity field $\mathbf{u}$ around C (traversed anticlockwise) is $-2\sqrt{3}\,aU$. [3]

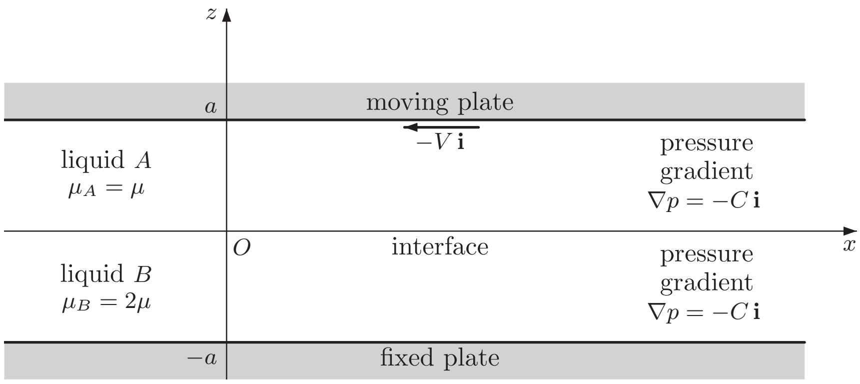
(g) Consider the particles making up the curve C (as in part (f)) at time $t = 0$. At a later time t, these same particles make up the curve $C(t)$, and it is found that

$$\oint_{C(t)} \mathbf{u} \cdot d\mathbf{r} \neq -2\sqrt{3}\,aU.$$

What can be deduced from this result? [2]

Question 4 (*Unit 8*)

Consider the steady, laminar flow of two liquids, A and B, between infinite parallel plates at $z = \pm a$, as shown in the diagram below. The plate at $z = -a$ is fixed, while the plate at $z = a$ moves with constant velocity $-V\,\mathbf{i}$, where $V > 0$. The liquids do not mix, and each forms a layer of depth a. There is an applied pressure gradient acting on both liquids, given by $\boldsymbol{\nabla} p = -C\,\mathbf{i}$ (where $C > 0$ is constant), and the effects of gravitation are negligible.



(a) Assuming that each fluid is of constant viscosity and is Newtonian, what other assumption must be made about the fluids in order to apply Equations (2.8) on page 179 of *Unit 8*? Given the problem statement, why is this additional assumption reasonable? [*2*]

(b) The following assumptions are to be made. Write down the mathematical consequences of each of them.

1. The flow is two-dimensional.
2. There is no variation in the direction into the page.
3. The flow is steady.
4. There is no variation of velocity parallel to the plates.

Hence write down the continuity equation for either of the two liquids. [*3*]

(c) Show that the fluid velocities $\mathbf{u}_A$ in liquid A and $\mathbf{u}_B$ in liquid B are given by

$$\mathbf{u}_A = u_A(z)\,\mathbf{i} \qquad \text{and} \qquad \mathbf{u}_B = u_B(z)\,\mathbf{i}.$$

State the boundary conditions (four in all) at the upper and lower plates and at the interface $z = 0$. (Note that the shear stress must vary continuously between the plates.) [*6*]

(d) Write down the x-components of the Navier–Stokes equations for u_A and u_B. Solve them, using the boundary conditions from part (c), to show that

$$u_A(z) = -\frac{C}{6\mu}\left(3z^2 + (4\gamma - 1)\,az + 2(\gamma - 1)\,a^2\right),$$

$$u_B(z) = -\frac{C}{12\mu}\left(3z^2 + (4\gamma - 1)\,az + 4(\gamma - 1)\,a^2\right), \quad \text{where } \gamma = \frac{\mu V}{Ca^2}.$$ [*12*]

(e) For the case $\gamma = \frac{1}{2}$, sketch the velocity profile of the flow, indicating the fluid velocity at the interface in terms of C, a and μ. [*2*]