UNIVERSITY OF SURREY©

Faculty of Engineering and Physical Sciences

Department of Mathematics

Undergraduate Programmes in Mathematical Studies

Module MAT1003 — 10 Credits

REAL ANALYSIS I - RESIT COURSEWORK

Level HE1 Examination

Time allowed: Hand-in date August 19 2011 Semester 1, 2010/11

Resit Coursework - Attempt all questions and justify your answers

Calculators and class notes are allowed.

Additional material:
None

Question 1

(a) Prove that $\frac{1}{2}\sqrt{5}$ is irrational. [3]

(b) Assume that x is algebraic and let $y = x + 3$. Is y algebraic? Explain your answer. [4]

(c) Does any open interval in $\mathbb{R}$ have a maximum? Explain your answer. [3]

Question 2

(a) Express in quantifiers:

$$\text{The sequence } \{a_n\}_{n\in\mathbb{N}} \text{ is strictly decreasing.} \qquad (*)$$

Give the negation of $(*)$ as well. [2]

(b) Find the sup, inf, min and max (or indicate that they don't exist) for the following sets and sequences:

(a) $[-1, 3) \cap (-\infty, 2)$.

(b) $\bigcap_{n\in\mathbb{N}} \left(\frac{1}{n}, 4 + \frac{1}{n}\right)$.

(c) $\left\{x \in \mathbb{Q} : x^3 \leq \pi\right\}$.

(d) $\left(n - 2\sqrt{n}\right)_{n\in\mathbb{N}}$. [5]

(c) Prove that a set has at most one maximum. Indicate the precise definitions and axioms you use in your argument. [3]

Question 3

(a) Let the sequence $(a_n)_{n\in\mathbb{N}}$ be defined recursively by $a_1 = 0$ and $a_{n+1} = \frac{1}{3-a_n}$ for $n \geq 1$. Give the first four terms of this sequence. [2]

(b) Prove by induction that $a_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$ and that $(a_n)_{n\in\mathbb{N}}$ is strictly increasing. [4]

(c) You may assume that $a_n \leq 2$ for all n. State the theorem that allows you to conclude that $(a_n)_{n\in\mathbb{N}}$ converges. [2]

(d) Can you conclude that $(a_n)_{n\in\mathbb{N}}$ converges in $\mathbb{Q}$? Explain your answer. [2]

Question 4

Circle the correct answer in the following five multiple choice questions.

(a) The convergence of which of the following series **cannot** be decided by the Leibniz test?

a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ b) $\sum_{n=1}^{\infty} \frac{\cos \pi n}{n}$ c) $\sum_{n=1}^{\infty} \frac{(-1)^n(n+2)}{2n}$ d) $\sum_{n=2}^{\infty} \frac{2}{(\ln \frac{1}{n})^n}$ [2]

(b) If $a \in \mathbb{Q}$ and $b \in \mathbb{R} \setminus \mathbb{Q}$, which of the following **can** be true?

a) $ab \in \mathbb{Q}$ b) $a + b \in \mathbb{Q}$ c) $a^2 \in \mathbb{Q}$ d) $\sqrt{b} \in \mathbb{Q}$ [2]

(c) What is the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$?

a) $1/e$ b) $-1 + 1/e$ c) 1 d) $-\ln 2$ [2]

(d) What is the limit of $\frac{n! + e^n}{n^n + n^5}$ as $n \to \infty$?

a) 0 b) 1 c) e d) the sequence diverges. [2]

(e) Let $(a_n)_{n\in\mathbb{N}}$ be an increasing sequence in $\mathbb{R}$. Which of the following is necessarily true?

a) $a_n \to \infty$

b) $\lim_{n\to\infty} a_n = \max\{a_n \mid n \in \mathbb{N}\}$.

c) $\lim_{n\to\infty} a_n$ exists.

d) If $(a_n)_{n\in\mathbb{N}}$ is bounded, then $\lim_{n\to\infty} a_n$ exists. [2]

Question 5

(a) Compute the sums of the following series.

$$(a) \quad \sum_{n=0}^{\infty} \frac{2^{n+1} + (-3)^n}{7^n} \qquad (b) \quad \sum_{n=1}^{\infty} \frac{3n+1}{n^2 + n^3}$$

[3]

(b) Give the region of convergence for the following series, and show your working.

$$(a) \quad \sum_{n=0}^{\infty} \frac{5^n}{n} x^{n+1} \qquad (b) \quad \sum_{n=1}^{\infty} \frac{1}{n^3} \left(\frac{x}{x+2}\right)^n$$

[4]

(c) Which power series would you apply (and how) to compute $\ln 7 - \ln 6$ by hand? How many terms do you need to get the answer accurate for at least three decimal places? [3]

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