1/ Find the roots of the equation

$$z^3 = -(4\sqrt{3}) + 4i,$$

giving your answers in the form $re^{i\theta}$, where $r > 0$ and $0 \leqslant \theta < 2\pi$.

Denoting these roots by z_1, z_2, z_3, show that, for every positive integer k,

$$z_1^{3k} + z_2^{3k} + z_3^{3k} = 3(2^{3k} e^{\frac{5}{6}k\pi i}).$$

2/ Write down an expression in terms of z and N for the sum of the series

$$\sum_{n=1}^{N} 2^{-n} z^n.$$

Use de Moivre's theorem to deduce that

$$\sum_{n=1}^{10} 2^{-n} \sin\left(\tfrac{1}{10}n\pi\right) = \frac{1025 \sin\left(\frac{1}{10}\pi\right)}{2560 - 2048 \cos\left(\frac{1}{10}\pi\right)}.$$

3/ Write down all the 8th roots of unity. [2]

Verify that

$$(z - e^{i\theta})(z - e^{-i\theta}) \equiv z^2 - (2\cos\theta)z + 1.$$ [1]

Hence express $z^8 - 1$ as the product of two linear factors and three quadratic factors, where all coefficients are real and expressed in a non-trigonometric form. [5]

$1, e^{\pi i/4}, e^{\pi i/2}, e^{3\pi i/4}, -1, e^{5\pi i/4}, e^{3\pi i/2}, e^{7\pi i/4}$

$(z^4 - 1)(z^4 + 1)$

$(z-1)(z+1)(z^2+1)$

$(z-1)(z+1)(z - e^{\pi i/4})(z - e^{-\pi i/4})$

$(z-i)(z+i)(z - e^{3\pi i/4})(z - e^{-3\pi i}$

$(z-1)(z+1)(z - z\sqrt{2} + 1)(z + z\sqrt{2} + 1)$

(z^2+1)