Calculus III, 2011: Coursework 3
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The deadline is 4PM on Thursday Oct 20th. Hand-in **Question 4 only** *to the green box on the basement floor of the Maths building.*

1. Sketch the curves whose parametric equations are

(a) $\mathbf{r} = (2\cos 2t)\mathbf{i} + (3\sin 2t)\mathbf{j}$

(b) $\mathbf{r} = 2t\mathbf{i} - t\mathbf{j} + t^2\mathbf{k}$

(c) $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$

and write down the derivatives $\mathrm{d}\mathbf{r}/\mathrm{d}t$ and $\mathrm{d}^2\mathbf{r}/\mathrm{d}t^2$ where they are defined.

2. A cardioid is defined by the polar equation $r = a(1+\cos\theta)$. Sketch the curve, and evaluate (a) the arc-length , and (b) the enclosed area, over one complete loop of the cardioid.

3. An ellipse is constructed via the "two pins and loop of string" method: defining one focus at the origin, and the second focus at $(x = -2c, y = 0)$, the ellipse C is defined as the locus of all points P satisfying $r_1 + r_2 = 2a$, where r_1, r_2 are the distances of P from the two foci (and $a > c$). Show that the polar equation of C is

$$r_1(\theta) = \frac{a(1-e^2)}{1+e\cos\theta} \quad ,$$

where $e = c/a$. (Hint: use the cosine rule for a suitable triangle). Show that the ellipse has semi-major axis a and semi-minor axis $b = a\sqrt{(1-e^2)}$.

4. (*) Hand-in question:
A cycloid is defined by the parametric equations $x = a(t - \sin t),\ y = a(1 - \cos t)$.

(a) Sketch the curve for $0 \le t \le 4\pi$. [3]

(b) Evaluate the arc-length of the curve for one arch with endpoints at $(0, 0)$ and $(2\pi a, 0)$. [4]

(c) Evaluate the area between the one arch in (b) and the x-axis, using $A = \int y(t)\,(dx/dt)\,dt$. [3]

5. Evaluate the arc-length of the parabola $y = x^2$ between $x = 0$ and $x = a$. (Hint: look up the integral in a table, e.g. Thomas T-1 number 21, or use a "sinh" substitution).

6. Find the relation of $\mathrm{d}v/\mathrm{d}t$ to $\mathrm{d}\mathbf{v}/\mathrm{d}t$ for any non-zero vector $\mathbf{v}(t)$. Hence show that

$$\mathrm{d}v/\mathrm{d}t = 0 \Rightarrow \mathrm{d}\mathbf{v}/\mathrm{d}t \perp \mathbf{v} \ .$$

[Note: The last sentence is true, for example, for motion in a circle.]