Calculus III, 2011: Coursework 3

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The deadline is 4PM on Thursday Oct 20th. Hand in **Question 4 only** *to the green box on the basement floor of the Maths building.*

1. Sketch the curves whose parametric equations are

(a) $\mathbf{r} = (2\cos 2t)\mathbf{i} + (3\sin 2t)\mathbf{j}$

(b) $\mathbf{r} = 2t\mathbf{i} - t\mathbf{j} + t^2\mathbf{k}$

(c) $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$

$(-\infty \le t \le \infty)$, and write down the derivatives $\mathrm{d}\mathbf{r}/\mathrm{d}t$ and $\mathrm{d}^2\mathbf{r}/\mathrm{d}t^2$ where they are defined.

Answers: (a) is an ellipse in the xy plane, semi-axes 2 and 3 units. (Note the 2 in $\sin 2t$ doesn't change the actual curve, it just makes the point $\mathbf{r}$ loop round the ellipse each time t increases by π, rather than 2π. If the coefficients of t were different in $\mathbf{i}$ and $\mathbf{j}$ this would not be true).

Derivatives are $d\mathbf{r}/dt = (-4\sin 2t)\mathbf{i} + (6\cos 2t)\mathbf{j}$, and $d^2\mathbf{r}/dt^2 = (-8\sin 2t)\mathbf{i} - (12\sin 2t)\mathbf{j}$.

(b) is a parabola, with the symmetry axis along $\mathbf{k}$ and in the plane containing $\mathbf{2i} - \mathbf{j}$. The derivatives are $d\mathbf{r}/dt = (2, -1, 2t)$ and $d^2\mathbf{r}/dt^2 = (0, 0, 2)$.

(c) is a helix with radius 1, and symmetry axis the z axis. The derivatives are $d\mathbf{r}/dt = (-\sin t, \cos t, 1)$ and $d^2\mathbf{r}/dt^2 = (-\cos t, -\sin t, 0)$.

2. A cardioid is defined by the polar equation $r = a(1+\cos\theta)$. Sketch the curve, and evaluate (a) the arc-length , and (b) the enclosed area, over one complete loop of the cardioid.

Answer: The curve is an "apple shape" with one cusp at $\theta = \pi$. The arc-length is

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{r^2 + \frac{dr}{d\theta}^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\, d\theta \\ &= a\int_0^{2\pi} \sqrt{2 + 2\cos\theta}\, d\theta \\ &= a\int_0^{2\pi} \sqrt{4\cos^2\frac{\theta}{2}}\, d\theta \end{aligned}$$

where we have used the half-angle formula (similar to the double-angle formula). Now we need to take the positive root, so the integrand is actually $2|\cos\frac{\theta}{2}|$; (beware: if you don't do that you will get zero !). This is simplified by symmetry if we just take the integral from 0 to π (where $\cos\frac{\theta}{2}$ is positive), and double it, so we get

$$\begin{aligned} L &= 4a\int_0^\pi \cos\tfrac{\theta}{2}\,d\theta \\ &= 4a\left[2\sin\tfrac{\theta}{2}\right]_0^\pi \\ &= 8a \end{aligned}$$

(b) The area enclosed is

$$\begin{aligned} A &= \int_0^{2\pi}\frac{1}{2}r^2\,d\theta \\ &= \frac{1}{2}a^2\int_0^{2\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta \\ &= \frac{1}{2}a^2\int_0^{2\pi}(1+2\cos\theta+\tfrac{1}{2}(1+\cos 2\theta))\,d\theta \\ &= \frac{1}{2}a^2\left[\tfrac{3}{2}\theta+2\sin\theta+\tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} \\ &= \frac{3\pi}{2}a^2 \end{aligned}$$

3. An ellipse is constructed via the "two pins and loop of string" method: defining one focus at the origin, and the second focus at $(x=-2c, y=0)$, the ellipse C is defined as the locus of all points P such that $r_1+r_2=2a$, where r_1, r_2 are the distances of P from the two foci (and $a>c$). Show that the polar equation of C is

$$r_1(\theta)=\frac{a(1-e^2)}{1+e\cos\theta} \quad ,$$

where $e=c/a$. (Hint: use the cosine rule for a suitable triangle). Show that the ellipse has semi-major axis a and semi-minor axis $b=a\sqrt{(1-e^2)}$.

Answer: Let the foci be labelled F_1, F_2; Consider a general point P on the ellipse at (r_1,θ). The cosine rule applied to triangle F_2, F_1, P gives

$$r_2^2=(2c)^2+r_1^2-2r_1(2c)\cos(\pi-\theta)$$

since F_2 is on the negative x-axis, so the included angle $F_2\,F_1\,P$ is $\pi-\theta$. Substituting in $r_2=2a-r_1$ from the sum-of-distances given, we have

$$4a^2-4ar_1+r_1^2=4c^2+r_1^2+4cr_1\cos\theta\ .$$

Now the r_1^2 cancels, and we rearrange to

$$r_1(a+c\cos\theta)=a^2-c^2 \quad ;$$

and from definitions, $e = c/a$, so insert $c = ea$ and then

$$r_1 = \frac{a(1-e^2)}{1+e\cos\theta}\,. \qquad \text{QED}$$

Here the top is a constant (often called ℓ), so the min and max values of r_1 clearly occur at max/min values of the bottom, i.e. $\theta = 0, \pi$; so the ellipse crosses the x-axis at $x_1 = a(1-e^2)/(1+e)$ and $x_2 = -a(1-e^2)/(1-e)$. Then $x_1 - x_2 = 2a$ is the major axis length, and a is the semi-major axis.

For the semi-minor axis, we want the max/min values of y on the ellipse. We have $y = r_1 \sin\theta$ from the usual conversion from polar to Cartesian coords. This y is max/min when $dy/d\theta = 0$; solving for that, we find $\cos\theta = -e$ and so $\sin\theta = \pm\sqrt{1-e^2}$; hence the semi-minor axis is $b = a\sqrt{1-e^2}$.

4. (*) Hand-in question:

A cycloid is defined by the parametric equations $x = a(t - \sin t),\ y = a(1-\cos t)$.

(a) Sketch the curve for $0 < t < 4\pi$. [3]

(b) Evaluate the arc-length of the curve for one arch with endpoints at $(0,0)$ and $(2\pi a, 0)$. [4]

(c) Evaluate the area between the one arch in (b) and the x-axis, using $A = \int y(t)(dx/dt)dt$. [3]

Answer: First note that y is bounded by $0 < y < 2a$, and $y = 0$ when $\cos t = +1$ i.e. $t = 0, 2\pi, 4\pi, \ldots$. At these points, we get $x = 0, 2\pi a, 4\pi a, \ldots$. The maximum y is $y = 2a$ at $t = \pi, 3\pi, \ldots$; so over all t the curve has an infinite series of identical 'arches'. Over the given range $0 < t < 4\pi$ we get two arches.

(b) For the arc-length, we use the formula for length of a parametric curve, and we get

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int \sqrt{(a - a\cos t)^2 + (a\sin t)^2}\, dt \\
&= a\int \sqrt{(2 - 2\cos t)}\, dt \\
&= a\int \sqrt{4\sin^2 \tfrac{t}{2}}\, dt \\
&= a\int_0^{2\pi} 2\sin\tfrac{t}{2}\, dt \\
&= a\left[-4\cos\tfrac{t}{2}\right]_0^{2\pi} \\
&= 8a
\end{aligned}$$

Note that for the limits on t, we have to solve for the t values at the given (x, y) endpoints. From part (a), these are clearly seen to be $t = 0$ and $t = 2\pi$.

(c) For the area, from the given formula we have

$$\begin{aligned} A &= \int_0^{2\pi} a(1-\cos t)a(1-\cos t)\ dt \\ &= a^2 \int 1 - 2\cos t + \cos^2 t\ dt \\ &= a^2 \int 1 - 2\cos t + \tfrac{1}{2}(1+\cos 2t)\ dt \\ &= a^2 \left[\tfrac{3t}{2} - 2\sin t + \tfrac{1}{4}\sin 2t\right]_0^{2\pi} \\ &= 3\pi a^2 \end{aligned}$$

(Note that both of these answers look quite similar to Q2; the reason is that a cardioid can also be generated by a smaller circle rolling inside a larger circle of twice the radius).

5. Evaluate the arc-length of the parabola $y = x^2$ between $x = 0$ and $x = a$. (Hint: look up the integral in a table, e.g. Thomas T-1 number 21, or use a "sinh" substitution).

Answer: The integral we need is

$$\begin{aligned} L &= \int_0^a \sqrt{1 + (dy/dx)^2}\ dx \\ &= \int_0^a \sqrt{1+4x^2}\ dx \\ &= \int_0^a 2\sqrt{\tfrac{1}{4} + x^2}\ dx \end{aligned}$$

This looks a bit tricky; but using a substitution $x = \frac{1}{2}\sinh u$, we get $dx/du = \frac{1}{2}\cosh u$ and the integral becomes

$$\begin{aligned} L &= \int_0^b 2\sqrt{\tfrac{1}{4}(1+\sinh^2 u)}\frac{dx}{du}du \\ &= \int_0^b \tfrac{1}{2}\cosh^2 u\ du \\ &= \int_0^b \tfrac{1}{4}(1+\cosh 2u)\ du \\ &= \left[\tfrac{1}{4}u + \tfrac{1}{8}\sinh 2u\right]_0^b \\ &= \left[\tfrac{1}{4}u + \tfrac{1}{4}\sinh u\cosh u\right]_0^b \\ &= \tfrac{1}{4}(b + \sinh b\cosh b) \end{aligned}$$

where $b = \sinh^{-1}(2a)$ is the upper limit on u; finally we insert values $b = \ln(2a+\sqrt{4a^2+1})$, $\sinh b = 2a$, $\cosh b = \sqrt{1+(2a)^2}$, and finally we get

$$L = \tfrac{1}{4}\left[\ln(2a+\sqrt{1+4a^2}) + 2a\sqrt{1+4a^2}\right] \quad .$$

As a check, this tends to a for $a \ll 1$ and a^2 for $a \gg 1$, as expected from a sketch.

6. Find the relation of $\mathrm{d}v/\mathrm{d}t$ to $\mathrm{d}\mathbf{v}/\mathrm{d}t$ for any non-zero vector $\mathbf{v}(t)$. Hence show that

$$\mathrm{d}v/\mathrm{d}t = 0 \Rightarrow \mathrm{d}\mathbf{v}/\mathrm{d}t \perp \mathbf{v}\ .$$

[Note: The last sentence is true, for example, for motion in a circle.]

Answer: Since $v^2 = \mathbf{v} \cdot \mathbf{v}$, differentiating both sides with respect to t we have

$$2v\frac{\mathrm{d}v}{\mathrm{d}t} = 2\mathbf{v} \cdot \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\ .$$

Dividing by $2v$, we get

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \hat{\mathbf{v}} \cdot \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\ ,$$

where $\hat{\mathbf{v}}$ is the unit vector. From the usual dot-product rule, this evaluates to $|d\mathbf{v}/dt| \cos\theta$, where θ is the angle between $d\mathbf{v}/dt$ and $\mathbf{v}$.

If the LHS is zero we must have

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \cdot \mathbf{v} = 0$$

which implies the required result (two vectors are perpendicular if their dot product is zero).

Note: here and in many places in this course, it is very important to keep clear both in your notation, and in your mind, when you have a scalar and when you have a vector. Be careful not to lose the dot in a dot product or a divergence, or the cross in a cross product or a curl. It is probably sensible to avoid using dot or cross for multiplication by a scalar, to avoid possible confusions.