Calculus III, 2011: Coursework 1 Answers
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The deadline is 4PM on Thursday Oct 6th. Hand-in is to the green box on the basement floor of the Maths building. Hand in the solution for **Question 3 only.**

1. (a) Using the fact that $1/12 = 1/4 - 1/6$, find $\sin\frac{\pi}{12}$,

(b) and use the double angle formulae to find $\cos^2\frac{\pi}{8}$.

Answer:

(a) $\sin(\pi/12) = \sin(\pi/4)\cos(-\pi/6) + \cos(\pi/4)\sin(-\pi/6) = \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\frac{-1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$

(b) $\cos^2(\pi/8) = \frac{1}{2}(1+\cos(2\pi/8)) = \frac{2+\sqrt{2}}{4}$.

2. Find $\int \sin^4 x\,dx$; hence evaluate $\int_0^{2\pi}\sin^4 x\,dx$.

Answer: We use the double-angle formula for $\sin^2 x, \cos^2 x$ twice here:

$$\begin{aligned}
\sin^2 x &= \frac{1}{2}(1-\cos 2x)\\
\sin^4 x &= \frac{1}{4}(1-2\cos 2x+\cos^2 2x)\\
&= \frac{1}{4}(1-2\cos 2x+\frac{1}{2}(1+\cos 4x))\\
&= \frac{3}{8}-\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x\ .\\
\text{so}\int \sin^4 x\,dx &= \frac{3}{8}x-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+\text{const.}\\
\text{and}\int_0^{2\pi}\sin^4 x\,dx &= \frac{3}{4}\pi.
\end{aligned}$$

It's worth remembering that the double-angle formulas, along with $\sin x\cos x = \frac{1}{2}\sin 2x$, can be used to turn any product like $\sin^m x\cos^n x$ into a sum.

3. (*) Hand-in:

(a) Prove that $\int_0^{2\pi}\sin mx\cos nx = 0$ for any positive integers m, n.

(b) Using vectors, calculate the acute angle between any two body diagonals of a cube (note, a body diagonal joins two opposite corners).

Answers:

(a) Using the product rule (Eq.1.11 from the notes), we get

$$\begin{aligned}
\int_0^{2\pi}\sin mx\,\cos nx\,dx &= \int_0^{2\pi}\frac{1}{2}\left[\sin(m+n)x+\sin(m-n)x\right]dx\\
&= \frac{1}{2}\left[\frac{-1}{m+n}\cos(m+n)x+\frac{-1}{m-n}\cos(m-n)x\right]_0^{2\pi} \qquad (\text{if } m\neq n)
\end{aligned}$$

Since $\cos(2\pi k) = 1$ for any integer k, all the cos terms above are 1, and the definite integral is zero for $m \neq n$.

Note the second line above does not work if $m = n$ because we have a divide-by-zero, so we treat $m = n$ as a separate case: go back to the first line and find $\sin(m-n)x = \sin 0x = 0$, so the second term vanishes before the integration, and the first term integrates to zero as before.

Now we've shown both cases $m \neq n$ and $m = n$ give zero, so we're done.

(b) Without loss of generality, to simplify, we choose one corner of our cube at the origin, let the side length be 1 and align the cube with the x, y, z axes. Then it is easy to see that the eight corners of our cube have position vectors
$\mathbf{0},\ \mathbf{i},\ \mathbf{j},\ \mathbf{k},\ \mathbf{i}+\mathbf{j},\ \mathbf{i}+\mathbf{k},\ \mathbf{j}+\mathbf{k},\ \mathbf{i}+\mathbf{j}+\mathbf{k}$.
(i.e. all the coordinates x, y, z are either 0 or 1, and there are $2^3 = 8$ ways to do this.)

Since the vector distance from point P to Q is $\mathbf{q} - \mathbf{p}$, one body diagonal is $\mathbf{a} = \mathbf{i}+\mathbf{j}+\mathbf{k}-\mathbf{0} = (1,1,1)$, and another is (for example) $\mathbf{b} = (\mathbf{i}+\mathbf{j}) - \mathbf{k} = (1,1,-1)$ (a sketch will help here).

We want the angle between these, use $\cos\theta = \mathbf{a}.\mathbf{b}/|\mathbf{a}|\,|\mathbf{b}|$: we have $|\mathbf{a}| = |\mathbf{b}| = \sqrt{3}$, and the scalar product $\mathbf{a}.\mathbf{b} = +1$. Hence the acute angle is $\theta = \arccos(\frac{1}{3}) = 1.23$ rad or 70.5 deg.

Note choosing one of the above diagonals in the opposite direction will get result -1 for the scalar product, which leads to angle $\arccos(-\frac{1}{3})$; this is the obtuse angle $\pi - \theta$ since $\cos(\pi - \theta) = -\cos\theta$, so subtract that from π to get the acute angle above) .

Note if you found a vector with length $\sqrt{2}$ instead of $\sqrt{3}$, then it is a face diagonal (joining two opposite corners on a common face), not a body diagonal.

4. Describe each of the following surfaces in three dimensional space. [For example, a suitable description of $z = 0$ is 'a plane perpendicular to the z axis' while $x^2 + y^2 = 1$ is 'a circular cylinder of radius 1 whose axis is the z axis'.]

(a) $x^2 + 4y^2 + 9z^2 - 36 = 0$

(b) $x^2 + y^2 - 9z^2 + 36 = 0$

(c) $x^2 + y^2 - 9z + 36 = 0$

[You will find many more examples in Thomas Exercises 12.6. Use them for practice: something like this was on the exams in 2007 and 2008, and has not been well attempted.]

Answers:

(i) Dividing by 36, we get

$$\frac{x^2}{6^2} + \frac{y^2}{3^2} + \frac{z^2}{2^2} = 1 \tag{1}$$

hence the surface is an ellipsoid, with centroid at the origin, with semi-major axes 6,3,2 units in x, y, z respectively.

(ii) We can rearrange into

$$\frac{x^2}{6^2} + \frac{y^2}{6^2} = \frac{z^2}{2^2} - 1$$

Consider various cut-planes $z = d$ for various constant d. This has no solutions for $-2 < d < 2$, and cuts the surface in a circle centred on the z-axis for $d < -2$ or $d > 2$. Intersection with a cut-plane $x =$ const or $y =$ const. will give a hyperbola. Therefore, the surface is a circular hyperboloid of two sheets. (special case of elliptic hyperboloid).

(iii) We can rearrange this into

$$\frac{x^2}{6^2} + \frac{y^2}{6^2} = \frac{z}{4} - 1$$

Again considering cut-planes $z = d$ for various constant d: there is no solution if $d < 4$, and the plane cuts our surface in a circle if $d > 4$; radii increasing with d. Since z appears linearly (not as z^2), this surface is a circular paraboloid, with axis the z axis, and pole at $(0, 0, 4)$.

5. (Exercise 1.2 of notes). Evaluate, with explanation of the limits, the volume integral

$$\int_V e^x \, dx \, dy \, dz$$

where V is the tetrahedron bounded by the four planes $x = 0, y = 0, z = 0$ and $x + y + z = a$, $(a > 0)$.

Answer: Each face of the tetrahedron is part of one of the above planes: each edge is in two of the planes, and each corner of the tetrahedron is a point where three of our four planes meet: it is clear that one of these corners is the origin and the others are at $(a, 0, 0), (0, a, 0), (0, 0, a)$, where the tilted plane $x + y + z = a$ crosses two of the $x = 0, y = 0, z = 0$ planes.

Next, the volume integral runs over all the **interior** of that tetrahedron, so the interior is the set of all points with $x, y, z \geq 0$ and $x+y+z \leq a$. So, if we take z as the outermost integral, we need limits $0 < z < a$. Now we need to work out the x and y limits at each z: At a fixed z we need $x + y \leq a - z$, so now we have a triangular area: so we need $0 < y < a - z$ and then $0 < x < a - y - z$.

Hence we have

$$\begin{aligned}
I &= \int_{z=0}^{a} \int_{y=0}^{a-z} \left(\int_{x=0}^{a-y-z} e^x \, dx \right) dy \, dz \\
&= \int_{z=0}^{a} \left(\int_{y=0}^{a-z} (e^{(a-y-z)} - 1) \, dy \right) dz \\
&= \int_{z=0}^{a} \left[-e^{(a-y-z)} - y \right]_{y=0}^{y=a-z} dz \\
&= \int_{z=0}^{a} (-e^0 - (a - z)) - (-e^{(a-z)} - 0) \, dz
\end{aligned}$$

$$
\begin{aligned}
&= \int_{z=0}^{a} -1 - a + z + e^{(a-z)}\, dz \\
&= \left[-z - az + \tfrac{1}{2}z^2 - e^{(a-z)}\right]_{z=0}^{a} \\
&= e^a - \tfrac{1}{2}a^2 - a - 1 \qquad .
\end{aligned}
$$

Note: a useful check of this is from the well-known Taylor series $e^a = 1 + a + a^2/2 + a^3/6 + \ldots$; we see the answer above becomes $a^3/6 + a^4/24 + \ldots$.

If we choose $a \ll 1$, the integrand e^x is ≈ 1 (plus a small number) so the answer must approximate the volume of our tetrahedron, which is (1/3) (Area of base)$\times$(Height) $= a^3/6$; so this is physically sensible.