3) a) Power series solution 4

exists since $p(x)=1 \neq 0$

series exists for all $x$ - radius of convergence is infinite.

b) $y=\sum_{n=0}^{\infty} a_n x^n \Rightarrow \frac{dy}{dx}=\sum_{n=1}^{\infty} a_n n x^{n-1}$

$\Rightarrow \frac{d^2y}{dx^2}=\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$

Sub into differential eq.

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - x\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

rebase both summations to form $\sum_{n=1} a_n x^n$

$$\sum_{n=1}^{\infty} a_{n+1}(n+1)n\, x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

Make 1 summation

$$\sum_{n=1}^{\infty} (a_{n+1}(n+1)n - a_{n-1})x^n = 0$$

True for all $x^n$ so $a_{n+1}(n+1)n - a_{n-1} = 0$