Question 2 (Unit 14)

Two incompressible inviscid fluids occupy the regions $0 < z < \infty$ and $-h < z < 0$ respectively, where z is measured vertically upwards. The upper fluid has density ρ_1 and the lower fluid, bounded by a fixed floor at $z = -h$, has density ρ_2, where $\rho_2 > \rho_1$. In this problem you are asked to investigate the propagation of small-amplitude waves within the two fluids.

(i) The velocity potentials for the upper and lower fluids are ϕ_1 and ϕ_2, respectively, $z = \zeta(x,t)$ is the displacement (from $z = 0$) of the interface, u_3 is the fluid velocity component parallel to the z-axis, and g is the magnitude of the acceleration due to gravity. Show that, under suitable modelling assumptions which should be stated clearly:

(a) $\dfrac{\partial^2 \phi_1}{\partial x^2} + \dfrac{\partial^2 \phi_1}{\partial z^2} = 0 \quad (0 < z < \infty)$,

$\lim\limits_{z\to\infty} u_3 = \lim\limits_{z\to\infty} \dfrac{\partial \phi_1}{\partial z} = 0$;

(b) $\dfrac{\partial^2 \phi_2}{\partial x^2} + \dfrac{\partial^2 \phi_2}{\partial z^2} = 0 \quad (-h < z < 0)$,

$u_3 = \dfrac{\partial \phi_2}{\partial z} = 0$ at $z = -h$;

(c) to first order, $\dfrac{\partial \zeta}{\partial t} = \dfrac{\partial \phi_1}{\partial z} = \dfrac{\partial \phi_2}{\partial z}$ at $z = 0$;

(d) to first order, $\rho_1 \left(\dfrac{\partial^2 \phi_1}{\partial t^2} + g \dfrac{\partial \phi_1}{\partial z} \right) = \rho_2 \left(\dfrac{\partial^2 \phi_2}{\partial t^2} + g \dfrac{\partial \phi_2}{\partial z} \right)$ at $z = 0$. [15]

(ii) Show that for the propagation of simple sinusoidal waves, the (scaled) wave number k and angular frequency ω satisfy the equation

$$\omega^2 = \frac{(\rho_2 - \rho_1) g k \sinh kh}{\rho_2 \cosh kh + \rho_1 \sinh kh}.$$ [8]

(iii) Hence find the wave condition relating the wave speed c and the wavelength λ. [2]

Question 3 (Revision)

(i) Determine the parts of the strip $|x| \le 2$, $-\infty < y < \infty$, in which the equation

$$y^2 \frac{\partial^2 u}{\partial x^2} + xy \frac{\partial^2 u}{\partial x \partial y} + \frac{1}{4} \frac{\partial^2 u}{\partial y^2} = 0$$

is elliptic, hyperbolic or parabolic. Indicate these parts (of the strip) and the associated classifications clearly on a diagram.

Indicate which parts of the strip, if any, have no real characteristics through each point. [7]

(ii) Consider the problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = x - 1 \quad (0 < x < 1,\ 0 < y < 1),$$
$$u(0,y) = u(1,y) = 0 \quad (0 < y < 1),$$
$$u(x,0) = u(x,1) = 0 \quad (0 < x < 1).$$

Specify a new dependent variable $v(x,y)$ which differs from $u(x,y)$ by a polynomial in x such that the original problem, when expressed in terms of v, becomes a problem involving a homogeneous differential equation. Write down this problem.

Describe each step in a procedure to solve this new problem and hence to find u (but do *not* carry out this procedure). [13]