This assignment covers *Units 5* to *8*. Each question is allotted 20 marks.

Question 1 (Unit 5)

Consider the two vector fields

$$\mathbf{a} = \boldsymbol{\omega} \times \mathbf{r}, \quad \text{where } \boldsymbol{\omega} = \omega(t)\mathbf{k} \text{ and } \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \text{ is the position vector,}$$

and

$$\mathbf{b} = c_1 x\mathbf{i} + c_2 y\mathbf{j}, \quad \text{where } c_1 \text{ and } c_2 \text{ are constant scalars such that } c_1 \neq -c_2.$$

(i) Which of these two vector fields can be used to model the velocity vector field of an incompressible flow? [2]

(ii) Which of these two vector fields can be used to model the velocity vector field of an irrotational flow? [2]

(iii) Write down the equations describing the stream function for the velocity vector field $\mathbf{a}$. Hence find the stream function for this flow, describe the form of the streamlines, and sketch some streamlines. Show by arrows the direction of the flow. [5]

(iv) Write down the equations describing the pathlines for the velocity vector field $\mathbf{b}$. Hence find an equation for y in terms of x that describes the pathlines. [3]

(v) An incompressible, inviscid liquid is contained within a closed circular cylinder and is rotating as a rigid body about the axis of the cylinder with constant angular speed Ω, so that the velocity vector field is $\mathbf{u} = \boldsymbol{\Omega} \times \mathbf{r}$, where $\boldsymbol{\Omega} = \Omega\mathbf{k}$. The liquid is subject to an external body force per unit mass

$$\mathbf{F} = (\alpha x + \beta y)\,\mathbf{i} + (\beta x + \alpha y)\,\mathbf{j},$$

where α and β are constants and the axis of symmetry of the cylinder is the z-axis. Write down Euler's equation of motion in component form for this flow, and show that the pressure p is given by

$$p(x, y) = \frac{\rho}{2}(\Omega^2 + \alpha)(x^2 + y^2) + \rho\beta xy + A,$$

where ρ is the density of the liquid and A is an arbitrary constant. [8]