Question 4 (Unit 8)

An incompressible, Newtonian, viscous fluid (constant coefficient of viscosity μ and density ρ) flows steadily in the annular region between two concentric, very long, circular cylinders of constant radii a and b ($b > a$). The outer cylinder is rotating with constant angular speed Ω about its axis, and the inner cylinder is maintained at rest. There are no body forces acting on the fluid. With reference to a cylindrical polar coordinate system $(r, theta, z)$ with the z-axis along the common axis, the velocity vector field $\mathbf{u}$ is independent of z and the flow is symmetrical about the z-axis.

(i) Write down the mathematical consequences of each of the following parts of the problem statement.

1. The fluid is incompressible.
2. The fluid flows steadily.
3. The velocity $\mathbf{u}$ is independent of z.
4. The flow is symmetrical about the z-axis.

Write down the boundary conditions on $\mathbf{u}$ at the cylinders $r = a$ and $r = b$. [4]

(ii) Use the continuity equation and part (i) to show that

$$\mathbf{u} = u_\theta \mathbf{e}_\theta + u_z \mathbf{e}_z,$$

where u_θ and u_z depend only on r. Here $\mathbf{e}_\theta$ and $\mathbf{e}_z$ are the conventional cylindrical polar unit vectors. [3]

(iii) Use the three components of the Navier–Stokes equation and parts (i) and (ii) to show that the pressure, p, satisfies

$$p = \int \rho \frac{u_\theta^2}{r}\, dr + f(z),$$

where $f(z)$ is an arbitrary function of z, and that u_θ and u_z satisfy

$$r^2 \frac{d^2 u_\theta}{dr^2} + r\frac{du_\theta}{dr} - u_\theta = 0$$

and

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du_z}{dr}\right) = \frac{1}{\mu}\frac{\partial p}{\partial z},$$

respectively. [5]

(iv) Hence show that the pressure gradient in the z-direction is constant, $-K$ say, and that

$$u_\theta = \frac{\Omega b^2}{b^2 - a^2}\left(r - \frac{a^2}{r}\right) \quad \text{and} \quad u_z = \frac{-Kr^2}{4\mu} + C\log_e r + D,$$

where

$$C = \frac{K(b^2 - a^2)}{4\mu \log_e(b/a)} \quad \text{and} \quad D = \frac{K(a^2 \log_e b - b^2 \log_e a)}{4\mu \log_e(b/a)}.$$ [8]