Question 3 (Unit 7)

(i) At a particular instant, some particles of a fluid of constant density constitute the perimeter of a triangle OAB, where O is the origin, A is the point $(1,0,0)$ and B is the point $(1,1,0)$ in Cartesian coordinates. The line OB is therefore given by $y = x$, $z = 0$. The velocity vector $\mathbf{u}$ is given by

$$\mathbf{u} = (x + y + z)\mathbf{i} + (x - y + 2z)\mathbf{j} + (x + 2y)\mathbf{k}.$$

(a) Using the definition of circulation, show by evaluating the line integral that the circulation of the velocity around the triangle OAB is zero.

(b) Comment on the validity of the following statement, paying attention to the conditions under which Kelvin's Theorem holds.

'As the fluid flows, the circuit OAB deforms, and it is possible to deduce from Kelvin's Theorem that the circulation of the velocity $\mathbf{u}$ around the deformed circuit is zero at all subsequent times.'

(c) Show that the circulation of the velocity $\mathbf{u}$ around every circuit in the fluid is zero. [8]

(ii) A tornado is modelled as a superposition of a line vortex and a sink. The line vortex has strength k and the sink has strength m, where k and m are both constants. The respective stream functions are

$$\psi_1 = \frac{-k}{2\pi}\log_e r \quad \text{and} \quad \psi_2 = \frac{-m\theta}{2\pi}$$

in cylindrical polar coordinates (r, θ, z).

(a) Find the velocity components of the flow. Show that the vorticity is zero everywhere except at $r = 0$. Assuming that the fluid is inviscid and that the only body force is gravity, find the pressure distribution $p = p(r, \theta, z)$ in the fluid in terms of m, k, the magnitude of the acceleration due to gravity g, and the (constant) fluid density ρ. (Your answer will contain an arbitrary constant.) [7]

(b) The above tornado model is not defined at $r = 0$. You are asked to *modify* the model so that it represents a tornado (with negligible vertical velocities) having:

(1) a core of fluid forming an essentially vertical cylindrical column in which the air rotates as a rigid body;

(2) similar features to the above model outside the core, e.g. the air is relatively still at large distances from the core.

Give the velocity vector field for this new model, both inside and outside the core. Show that there is now non-zero vorticity inside the core. [5]