Three perfect springs AB, BC, CD have stiffnesses k, $4k$, k respectively, and equal natural lengths l_0. Particles of equal mass m are attached to the springs at B and C, and the ends A and D are fixed to two points a horizontal distance $6l_0$ apart. The system is free to move in the line AD, as shown in the figure.

(i) Find the lengths of the springs when the particles are in equilibrium. [4]

(ii) Show that the displacements x_1 and x_2 of the particles at B and C from their equilibrium positions satisfy the differential equations

$$m\ddot{x}_1 = -5kx_1 + 4kx_2,$$
$$m\ddot{x}_2 = 4kx_1 - 5kx_2.$$

[6]

(iii) Find the normal mode angular frequencies for this system, and the corresponding normal mode displacement ratios. [9]

(iv) Write down expressions for the displacements of the two particles at time t after being released from arbitrary initial conditions. [3]

(v) The system is set in motion with particle B having an initial displacement from its equilibrium position of $\frac{1}{10}l_0$ towards the fixed point D, particle C having an initial displacement from its equilibrium position of $\frac{1}{10}l_0$ towards the fixed point A, and both particles initially at rest. Sketch a graph showing the motions of the two particles with time, indicating the most important features. [3]

Question 6 (*Unit 26*)

Note that the total number of marks available for this question is 25.

Consider the two vector fields

$$\mathbf{F}_1 = x^3\,\mathbf{i} + x^2y^2\,\mathbf{j} + xyz\,\mathbf{k},$$
$$\mathbf{F}_2 = y(2x+z)\,\mathbf{i} + x(x+z)\,\mathbf{j} + xy\,\mathbf{k}.$$

(i) By using a condition involving the curl of a vector field, which should be carefully stated, show that the vector field $\mathbf{F}_1$ is not conservative, whereas the vector field $\mathbf{F}_2$ is conservative. [7]

(ii) Evaluate the scalar line integral $\int_{C_1} \mathbf{F}_1 \cdot d\mathbf{r}$ of the vector field $\mathbf{F}_1$ along the curve C_1

$$x = t^2, \quad y = t, \quad z = t^3 \qquad (0 \le t \le 1)$$

from the point $(0,0,0)$ to the point $(1,1,1)$. [7]

(iii) Show that the scalar line integral $\int_{C_2} \mathbf{F}_2 \cdot d\mathbf{r}$ of the conservative vector field $\mathbf{F}_2$ along the curve C_2

$$x = at, \quad y = bt, \quad z = ct \qquad (0 \le t \le 1)$$

from the point $(0,0,0)$ to the point (a,b,c) takes the value

$$ab(a+c).$$

Hence find a scalar field $\phi(x,y,z)$ such that

$$\mathbf{F}_2 = \operatorname{grad}\phi \qquad \text{and} \qquad \phi(0,0,0) = 0.$$

[8]

(iv) By using the scalar field $\phi(x,y,z)$ from part (iii), evaluate the scalar line integral of the conservative field $\mathbf{F}_2$ along any curve from the point $(2,1,5)$ to the point $(-4,2,-3)$. [3]

PTO