Now, to derive the differential equation for $y(t)$, we let $v(t)$ be the speed of water leaving the bath through the hole and $V(t)$ be the volume of water left in the bath at time $t$. In a time interval $\delta t$ the amount of water leaving the hole is approximately $Sv(t)\delta t$, and this accounts for a *decrease* in the volume of water in the bath. Writing $\delta V$ as the *increase* in the volume of water in the bath in the time interval $\delta t$, we have

$$\delta V \simeq -Sv(t)\delta t, \tag{1}$$

i.e.

$$\frac{dV}{dt} = -Sv(t), \tag{2}$$

in the limit as $\delta t \to 0$. Since the bath is rectangular, $V = Ay$, so that (2) becomes

$$\frac{dy}{dt} = -\frac{Sv}{A}. \tag{3}$$

Various forms for $v(t)$ can be used; however, a simple, but realistic form that we shall use is due to Toricelli.

Toricelli's theorem says that the water particles leaving the hole attain the same speed as if they were released from rest and allowed to fall under the influence of gravity. Thus, applying conservation of mechanical energy to such a water particle of mass $m$, we have

$$\tfrac{1}{2}m \times 0^2 + mgy = \tfrac{1}{2}mv^2 + mg \times 0, \tag{4}$$

i.e.

$$v(t) = \sqrt{2gy(t)}. \tag{5}$$

The assumptions implicit in the form for $v$ in (5) are (a) there is no turbulence, and (b) there are no viscous, i.e. frictional, effects, especially at the hole. Combining (3) and (5) we obtain the following differential equation and initial condition for $y$:

$$\frac{dy}{dt} = -k\sqrt{y}, \quad y(0) = h, \tag{6}$$

where $k = \dfrac{S}{A}\sqrt{2g}$ is a constant. Equation (6) governs the variation of the depth of water in the bath with time $t$, and we shall now concentrate on the solution of (6).

The obvious approach for solving (6) is to separate the variables.

$$\int_h^0 \frac{1}{\sqrt{y}}\,dy = \int_0^t -k\,dt$$

Deliberate omission!

Thus

$$y = (\sqrt{h} - kt/2)^2, \tag{7}$$

where $k = \dfrac{S}{A}\sqrt{2g}$. Equation (7) informs us that the depth of water varies quadratically with time as shown in Fig. 2.

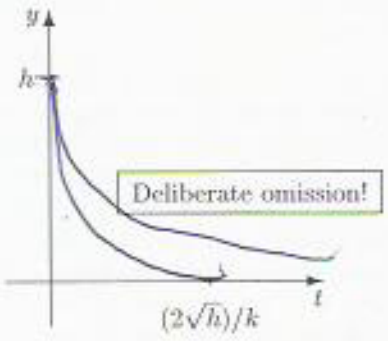


Fig. 2