(i) The Hamiltonian for $q > 0$ is $H(q,p) = \frac{1}{2}p^2 + aq$, whose contours are parabolas with the q-axis as axis of symmetry and with the vertex to the right. There is a reflecting barrier at the p-axis. Thus the phase curves are effectively closed, the motion is bound and therefore periodic.

The action variable is defined by the integral

$$I = \frac{\sqrt{2m}}{\pi}\int_0^{E/a} dq\,\sqrt{E-aq} = \frac{\sqrt{2m}}{\pi}\left[-\frac{2}{3a}(E-aq)^{3/2}\right]_0^{E/a} = \frac{2\sqrt{2m}}{3\pi a}E^{3/2},$$

which gives

$$E(I) = \left(\frac{3\pi a I}{2\sqrt{2m}}\right)^{2/3} = AI^{2/3}$$

and

$$\omega(I) = \frac{dE}{dI} = \tfrac{2}{3}AI^{-1/3}, \quad \text{where} \quad A = \left(\frac{3\pi a}{2\sqrt{2m}}\right)^{2/3}.$$

The angle variable may be chosen to be zero when $q = 0$; then the integral for $\theta(q)$ is

$$\begin{aligned}\theta &= \int_0^q dq\,\frac{\partial p}{\partial I} = \omega\int_0^q dq\,\frac{\partial p}{\partial E} = m\omega\int_0^q dq\,\frac{1}{p(q,E)}\\ &= \frac{m\omega}{\sqrt{2m}}\int_0^q \frac{1}{\sqrt{E-aq}}\\ &= \frac{\omega\sqrt{2mE}}{a}\left(1-\sqrt{1-\alpha q}\right), \quad \alpha = \frac{a}{E}, \quad 0 \le \theta \le \pi.\end{aligned}$$

The external factor can be simplified by substituting for the expression for $\omega(I)$ found above, but is simpler to note that when $q = E/a$, $\theta = \pi$, which gives the required result immediately. Then on rearranging we have $1 - \alpha q = (1-\theta/\pi)^2$, or

$$q = I^{2/3}\frac{A}{a}\left(\frac{2\theta}{\pi} - \frac{\theta^2}{\pi^2}\right), \qquad \text{for } 0 \le \theta \le \pi.$$

(ii) With this choice of the origin $q(\theta)$ must be an even function of θ, so only the even terms of the Fourier series are non-zero; that is, all the coefficients of $\sin n\theta$ are zero and we have

$$q(\theta, I) = Q_0(I) + \sum_{n=1}^{\infty} Q_n(I)\cos n\theta.$$

On using the relations given in Section 9 of the Handbook and the evenness of $q(\theta)$ we have

$$\begin{aligned}Q_0(I) &= \frac{1}{\pi}\int_0^{\pi} d\theta\, q(\theta, I)\\ Q_n(I) &= \frac{2}{\pi}\int_0^{\pi} d\theta\, q(\theta, I)\cos n\theta.\end{aligned}$$

The first integral is simply

$$Q_0(I) = \frac{E}{a\pi}\int_0^{\pi} d\theta\left(\frac{2\theta}{\pi} - \frac{\theta^2}{\pi^2}\right) = \frac{2A}{3a}I^{2/3}.$$

The second integral can be evaluated by substituting the expression for $q(\theta)$ and using the given integrals. Alternatively, note that $\sqrt{1-\alpha q} = 1 - \theta/\pi$, and so define $\varphi = \pi - \theta$ to write the integral in the form

$$\begin{aligned}Q_n(I) &= \frac{2E\cos n\pi}{a\pi}\int_0^{\pi} d\phi\left(1 - \frac{\phi^2}{\pi^2}\right)\cos n\phi\\ &= -\frac{2E\cos n\pi}{a\pi^3}\int_0^{\pi} d\phi\,\phi^2\cos n\phi, \quad n \ne 0,\\ &= -\frac{4E}{an^2\pi^2} = -\frac{4A}{an^2\pi^2}I^{2/3}. \qquad (3)\end{aligned}$$