(5)

4) a) $V(q) = \begin{cases} aq & q \geq 0 \\ \infty & q < 0 \end{cases}$

The Hamiltonian is given by

$H(q,p) = \frac{p^2}{2m} + aq \qquad q \geq 0$

$p^2 = 2m(E - aq) \Rightarrow p = \sqrt{2m(E-aq)}$

The roots for which $p = 0$ are $q = E/a$. There are no roots for $q > E/a$ and since $V(q) = \infty$ for $q < 0$ we have $q \geq 0$ so $0 \leq q \leq E/a$ $\therefore$ All motion is bound.

$\dot{q} = \frac{dq}{dt} = \frac{p}{m} = \sqrt{\frac{2}{m}}\sqrt{E - aq}$. This is autonomous for q $\therefore$ If we solve it and obtain a solut $q(t)$, $\bar{q}(t) = q(t+T)$ will also be a solution, where t is the period since $\frac{d\bar{q}}{dt} = \frac{dq}{dt}(t+T) = \dot{q}(q(t+T)) = \dot{q}(\bar{q}(t))$

The action is given by

$I = \frac{1}{2\pi}\oint p\,dq$

$= 2\frac{\sqrt{2m}}{2\pi}\int_0^{E/a}\sqrt{E-aq}\,dq = \frac{\sqrt{2m}}{\pi}\int_0^{E/a}\sqrt{E-aq}\,dq$

Sub $q = \frac{E}{a}\sin^2\theta$ then $q = 0 \Rightarrow \theta = 0$, $q = E/a \Rightarrow \theta = \pi/2$

$dq = \frac{2E}{a}\sin\theta\cos\theta\,d\theta$

$I = \frac{\sqrt{2m}}{\pi}\int_0^{\pi/2}\sqrt{E}\cos\theta\,\frac{2E}{a}\sin\theta\cos\theta\,d\theta$

$= \frac{2\sqrt{2m}}{\pi}\frac{2E^{3/2}}{a}\int_0^{\pi/2}\cos^2\theta\sin\theta\,d\theta$

$= \frac{2\sqrt{2m}\,E^{3/2}}{\pi a}\left[-\frac{\cos^3\theta}{3}\right]_0^{\pi/2}$

$= \frac{2\sqrt{2m}\,E^{3/2}}{\pi a}\left[-\frac{\cos^3(\frac{\pi}{2})}{3} - -\frac{\cos^3(0)}{3}\right]$

$= \frac{2\sqrt{2m}\,E^{3/2}}{\pi a}\left[-\frac{0}{3} + \frac{1}{3}\right] = \frac{2\sqrt{2m}\,E^{3/2}}{3\pi a}$