(a) The fixed points of the system occur where

$$x(x+\lambda) = x, \quad \text{or} \quad x(x+\lambda-1) = 0.$$

If $F(x) = x(x+\lambda)$ then

$$F^2(x) = x(x+\lambda)\{x(x+\lambda)+\lambda\},$$

and so the period 2 points are roots of

$$x(x+\lambda)\{x(x+\lambda)+\lambda\} = x \quad \Longrightarrow \quad x\{(x+\lambda)[x(x+\lambda)+\lambda]-1\} = 0.$$

The fixed points of $F(x)$ are also roots of this equation, which means that $x+\lambda-1$ must be a factor of the term in braces. The quickest way of obtaining the factorisation is probably to set $x+\lambda-1=y$, so that $x+\lambda=y+1$, and

$$\begin{aligned}(x+\lambda)[x(x+\lambda)+\lambda]-1 &= (y+1)[(y+1-\lambda)(y+1)+\lambda]-1\\ &= (y+1)[y^2+(2-\lambda)y+1]-1\\ &= y(y^2+(3-\lambda)y+(3-\lambda)).\end{aligned}$$

The period 2 points are therefore given by

$$\begin{aligned}y &= \tfrac{1}{2}\left\{\lambda-3\pm\sqrt{(3-\lambda)^2-4(3-\lambda)}\right\}\\ &= \tfrac{1}{2}\left\{\lambda-3\pm\sqrt{(\lambda-3)(\lambda+1)}\right\},\end{aligned}$$

or equivalently

$$x = \tfrac{1}{2}\left\{-(1+\lambda)\pm\sqrt{(\lambda-3)(\lambda+1)}\right\}.$$

Now $F'(x) = 2x+\lambda$, and to determine the stability of the period 2 orbit we must calculate $F'(a)F'(b)$ where a and b are the period 2 points. But

$$\begin{aligned}F'(a)F'(b) &= \left\{-1+\sqrt{(\lambda-3)(\lambda+1)}\right\}\left\{-1-\sqrt{(\lambda-3)(\lambda+1)}\right\}\\ &= 1-(\lambda-3)(\lambda+1)\\ &= 4+2\lambda-\lambda^2.\end{aligned}$$

The period 2 orbit becomes unstable, and period doubling takes place, when $F'(a)F'(b) = -1$, that is, when

$$4+2\lambda-\lambda^2 = -1 \quad \Longrightarrow \quad \lambda^2-2\lambda-5=0 \quad \Longrightarrow \quad \lambda = 1+\sqrt{6} \qquad (\text{since } \lambda > 0).$$

Alternatively: put $x_n = y_n + a$ and chose a to cast the system into the form $y_{n+1} = y_n^2 + c$, used in *Unit 12*, Section 12.3. We have

$$\begin{aligned}y_{n+1} &= y_n^2 + y_n(2a+\lambda) + a^2 + \lambda a - a\\ &= y_n^2 + \tfrac{1}{4}\lambda(2-\lambda) \qquad \text{if} \quad a = -\tfrac{1}{2}\lambda.\end{aligned}$$

But the system $y_{n+1} = y_n^2 + c$ period doubles from a period 2 to a period 4 orbit when $c = -\frac{5}{4}$, so the corresponding period doubling for the given system occurs for $\lambda(2-\lambda) = -5$, or $\lambda = 1+\sqrt{6}$ (since $\lambda > 0$).

(b)(i) The successive iterates of $\frac{4}{7}$ under the tent map are

$$\tfrac{6}{7},\quad \tfrac{2}{7},\quad \tfrac{4}{7},$$

so $\frac{4}{7}$ has period 3.

The successive iterates of $\frac{8}{15}$ under the tent map are

$$\tfrac{14}{15},\quad \tfrac{2}{15},\quad \tfrac{4}{15},\quad \tfrac{8}{15},$$

so $\frac{8}{15}$ has period 4.

The successive iterates of $\frac{16}{31}$ under the tent map are

$$\tfrac{30}{31},\quad \tfrac{2}{31},\quad \tfrac{4}{31},\quad \tfrac{8}{31},\quad \tfrac{16}{31},$$

so $\frac{16}{31}$ has period 5.

Note that all these numbers are of the form

$$\frac{2^{k-1}}{2^k-1}.$$