(iii) At the nth resonance the action, I_n, satisfies $n\omega(I_n) = \Omega$, whence

$$\tfrac{2}{3}AI_n^{-1/3} = \frac{\Omega}{n}, \quad \text{or} \quad I_n = \left(\frac{2An}{3\Omega}\right)^3 = \frac{\pi^2 a^2}{3m}\left(\frac{n}{\Omega}\right)^3.$$

If $n\omega(I) \simeq \Omega$ then for small values of F only the nth harmonic of $q(\theta)$ is important: the combination of this with the $\sin\Omega t$ term is

$$Q_n \cos n\theta \cos\Omega t = \tfrac{1}{2}Q_n\{\cos(n\theta+\Omega t) + \cos(n\theta - \Omega t)\}.$$

On ignoring the rapidly varying part of this, and all other rapidly varying terms, the Hamiltonian becomes

$$H(\theta, I, t) \simeq E(I) + \tfrac{1}{2}FQ_n(I)\cos(n\theta - \Omega t).$$

Now make the usual canonical transformation, $\phi = \theta - \Omega t/n$, $J = I$, to put this in the form

$$K(\phi, J) \simeq E(J) - \frac{\Omega J}{n} + \tfrac{1}{2}FQ_n(J)\cos n\phi.$$

Now expand about the resonant action I_n found above by putting $J = I_n + X$ and retaining only the lowest order terms. On using Equation 3 for $Q_n(I)$ the Hamiltonian becomes

$$K(\phi, J) \simeq -\frac{AX^2}{9I_n^{4/3}} - \frac{2AFI_n^{2/3}}{an^2\pi^2}\cos n\phi.$$

The fixed points of this system are at $X = 0$ and $\sin n\phi = 0$, with $n\phi = 0$ being unstable: so the separatrix energy is $E_s = -2AFI_n^{2/3}/(an^2\pi^2)$ and the equation of the separatrix is

$$\frac{X^2}{9I_n^{4/3}} = \frac{2FI_n^{2/3}}{an^2\pi^2}(1 - \cos n\phi).$$

The separatrix is widest when $\cos n\phi = -1$, so the maximum value of X is

$$\max(X) = \frac{6I_n\sqrt{F}}{n\pi\sqrt{a}}$$

and the separatrix width is

$$S_n = 2\max(X) = \frac{4a\pi n^2\sqrt{aF}}{m\Omega^3}.$$

(iv) The distance between adjacent island centres is approximately

$$\Delta I_n \simeq \frac{(an\pi)^2}{m\Omega^3},$$

so the resonance approximation is valid only if

$$\frac{(an\pi)^2}{m\Omega^3} > 2\max(X) = \frac{4a\pi n^2\sqrt{aF}}{m\Omega^3} \quad \text{or} \quad F < \frac{a\pi^2}{16}.$$