TMA MS323 04 Question 3: Specimen Solution

(i) Since we must consider the effect of changing t to $-t$ we examine Hamilton's equations, which are

$$\dot{q} = p \qquad \dot{p} = -V'(q).$$

Note that these are unchanged by the substitution of $-t$ for t and $-p$ for p (with q remaining the same); it follows that if $(q(t), p(t))$ is a solution of these equations so is $(q(-t), -p(-t))$.

The Hamiltonian is an even function of p, so its phase portrait is symmetric about the q-axis. In particular, if a phase curve crosses the q-axis it does so at right angles, except where the crossing is at a fixed point; furthermore the phase curve itself must then be symmetric about the q-axis.

The separatrix, $(q_s(t), p_s(t))$, passing through the fixed point $(q_A, 0)$, crosses the q-axis at only one other point, say $(q_0, 0)$. Choose the time origin to be when this crossing occurs, so that with respect to this time variable $(q_s(0), p_s(0)) = (q_0, 0)$. Consider now the curve $(q(t), p(t)) = (q_s(-t), -p_s(-t))$. We know that it is a solution of Hamilton's equations; moreover, when $t = 0$ we have $(q(0), p(0)) = (q_0, 0)$. Thus $(q(t), p(t))$ is a solution of Hamilton's equations which satisfies the same initial conditions as the separatrix; by the uniqueness theorem for differential equations it must therefore coincide with the separatrix. Thus

$$q_s(-t) = q_s(t) \qquad p_s(-t) = -p_s(t),$$

as required.

(ii) Use Melnikov's integral with $\mathbf{x} = (q, p)$,

$$G_1 = \frac{\partial H_0}{\partial p} = p, \quad G_2 = -\frac{\partial H_0}{\partial q} = -V'(q)$$

and $\mathbf{P} = (0, \sin \Omega t)$, in the notation of *Unit 13* Equation 13.21. First, change the time origin, by setting $t = t' + \delta$, so that for the motion on the separatrix $q_s(t')$ is an even function of t' and $p_s(t')$ an odd function, as in part (i). Then $\mathbf{P} = (0, \sin \Omega(t' + \delta))$. The most convenient form of Melnikov's integral is

$$\begin{aligned} M(t_0) &= \int_{-\infty}^{\infty} d\tau \, \{G_1(\tau)P_2(\tau, \tau + t_0) - G_2(\tau)P_1(\tau, \tau + t_0)\} \\ &= \int_{-\infty}^{\infty} d\tau \, p_s(\tau) \sin \Omega(\tau + t_0 + \delta) \\ &= \cos \Omega t_0' \int_{-\infty}^{\infty} d\tau \, p_s(\tau) \sin \Omega\tau + \sin \Omega t_0' \int_{-\infty}^{\infty} d\tau \, p_s(\tau) \cos \Omega\tau \end{aligned}$$

where $t_0' = t_0 + \delta$. But $p_s(\tau)$ is an odd function, and as $\cos \Omega\tau$ is even the integrand of the last integral is odd and the integral itself is zero. Let the value of the first integral be I_1, assumed to be non-zero, so

$$M(t_0) = I_1 \cos \Omega(t_0 + \delta).$$

Thus Melnikov's integral has simple zeros (at $t_0 = -\delta + (n + \frac{1}{2})\pi)/\Omega$). It follows that the stable and unstable manifolds cross transversally and a homoclinic tangle develops in the neighbourhood of the unperturbed separatrix; neighbouring orbits separate exponentially and the motion is, locally, chaotic.