TMA MS323 04 Question 1: Specimen Solution

(i) The angle-action variables for the unperturbed system are

$$q = \sqrt{2I}\sin\theta, \quad p = \sqrt{2I}\cos\theta, \quad I \geq 0,$$

so in this representation the Hamiltonian is

$$\begin{aligned}
H(\theta, I, t) &= I + \varepsilon I^2 \cos^2\theta \sin^2\theta + \varepsilon I \sin^2\theta \sin\Omega t \\
&= I + \tfrac{1}{4}\varepsilon I^2 \sin^2 2\theta + \varepsilon I \sin^2\theta \sin\Omega t \\
&= I + \tfrac{1}{8}\varepsilon I^2 + \tfrac{1}{4}\varepsilon I \sin(2\theta - \Omega t) \\
&\quad + \tfrac{1}{8}\varepsilon I\{4\sin\Omega t - I\cos 4\theta - 2\sin(2\theta + \Omega t)\}
\end{aligned}$$

where we have used the relations

$$\cos 2\theta = 1 - 2\sin^2\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta$$

and

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$

which can be found in the Handbook.

The unperturbed frequency is $\omega = 1$, so the only term that can vary slowly is $\sin(2\theta - \Omega t)$ and this only when $\Omega \simeq 2$. Thus there is a resonance at $\Omega = 2$.

(ii) In order to remove the slowly varying term we make the canonical transformation $\phi = \theta - \frac{1}{2}\Omega t$, $J = I$, with the generating function $F_2(J, \theta) = (\theta - \frac{1}{2}\Omega t)J$. In the new representation the Hamiltonian is

$$\begin{aligned}
K(\phi, J, t) &= J + \tfrac{1}{8}\varepsilon J^2 + \tfrac{1}{4}\varepsilon J \sin 2\phi \\
&\quad + \tfrac{1}{8}\varepsilon J\{4\sin\Omega t - J\cos(4\phi + 2\Omega t) - 2\sin(2\phi + 2\Omega t)\} + \frac{\partial F_2}{\partial t} \\
&= \left(1 - \tfrac{1}{2}\Omega\right)J + \tfrac{1}{8}\varepsilon J^2 + \tfrac{1}{4}\varepsilon J \sin 2\phi \\
&\quad + \tfrac{1}{8}\varepsilon J\{4\sin\Omega t - J\cos(4\phi + 2\Omega t) - 2\sin(2\phi + 2\Omega t)\}.
\end{aligned}$$

This Hamiltonian can be written in the form

$$K(\phi, J, t) = \left(1 - \tfrac{1}{2}\Omega\right)J + \tfrac{1}{8}\varepsilon J^2 + \tfrac{1}{4}\varepsilon J \sin 2\phi + \varepsilon K_1(\phi, J, t) \tag{1}$$

where

$$K_1(\phi, J, t) = \tfrac{1}{8}\varepsilon J\{4\sin\Omega t - J\cos(4\phi + 2\Omega t) - 2\sin(2\phi + 2\Omega t)\}.$$

For $\Omega \simeq 2$ the first three terms of Equation (1) are slowly varying and $K_1(\phi, J, t)$ oscillates relatively rapidly.

The slowly varying terms can be written in the form

$$\overline{K}(\phi, J) = \tfrac{1}{4}\varepsilon\left\{(\nu + \sin 2\phi)J + \tfrac{1}{2}J^2\right\}, \quad \text{with} \quad 1 - \tfrac{1}{2}\Omega = \tfrac{1}{4}\varepsilon\nu. \tag{2}$$

Since $J = I$ and $I \geq 0$ we require that $J \geq 0$.

(iii) The fixed points of the Hamiltonian $\overline{K}(\phi, J)$ given in Equation 2 are at the roots of

$$0 = \frac{\partial\overline{K}}{\partial\phi} = \tfrac{1}{2}\varepsilon J\cos 2\phi$$

$$0 = \frac{\partial\overline{K}}{\partial J} = \tfrac{1}{4}\varepsilon\{(\nu + \sin 2\phi) + J\}.$$

The first equation is satisfied if $\phi = -\frac{1}{4}\pi$. With this value of ϕ the second equation gives $J = 1 - \nu$, which is positive if $\nu < 1$, so there is a real fixed point here.

At this root the second derivatives are

$$\frac{\partial^2\overline{K}}{\partial\phi^2} = -\varepsilon J\sin 2\phi = \varepsilon(1 - \nu) > 0$$

$$\frac{\partial^2\overline{K}}{\partial\phi\partial J} = \tfrac{1}{2}\varepsilon\cos 2\phi = 0$$

$$\frac{\partial^2\overline{K}}{\partial J^2} = \tfrac{1}{4}\varepsilon > 0.$$