The successive iterates of this number are

$$\frac{2^k-2}{2^k-1},\quad \frac{2}{2^k-1},\quad \frac{2^2}{2^k-1},\quad \cdots \quad \frac{2^{k-1}}{2^k-1},$$

so this number has period k.

Alternatively: note that

$$\begin{aligned}\frac{2^{k-1}}{2^k-1} &= \tfrac{1}{2}(1-2^{-k})^{-1} = \tfrac{1}{2}(1+2^{-k}+2^{-2k}+\cdots)\\ &= 2^{-1}+2^{-(k+1)}+2^{-(2k+1)}+\cdots.\end{aligned}$$

Thus $2^{k-1}/(2^k-1)$ has the binary representation

$$\frac{2^{k-1}}{2^k-1} = 0.1\underbrace{00\ldots0}_{k-1}1\underbrace{00\ldots0}_{k-1}1\cdots$$

so has period k.

Hence $\frac{4}{7}$ ($k=3$) has period 3; $\frac{8}{15}$ ($k=4$) has period 4; and $\frac{16}{31}$ ($k=5$) has period 5. The number with period k is

$$\frac{2^{k-1}}{2^k-1}.$$

(ii) The kth iterate of the binary number $0.b_1b_2b_3\ldots$ depends on whether b_k is 0 or 1: if $b_k=0$ the kth iterate is $0.b_{k+1}b_{k+2}b_{k+3}\ldots$ while if $b_k=1$ the kth iterate is $0.b^*_{k+1}b^*_{k+2}b^*_{k+3}\ldots$ A number of the form $0.011\ldots$ could arise only if ξ_0 contained the digits 0011 or 1100 (as a consecutive group), while a number of the form $0.101\ldots$ could arise only if ξ_0 contained the digits 0101 or 1010. None of these is possible; the first two have two adjacent 1 s and the second two a single 0 sandwiched between two 1 s.

The given number is clearly not periodic, since it is not eventually repeating; its orbit is not dense as it does not enter the interval whose endpoints are 0.011 and $0.01111\ldots = 0.1$, nor does it enter the interval whose endpoints are 0.101 and $0.10111\ldots = 0.11$.

(c)(i) The Jacobian determinant of the map is

$$\begin{vmatrix} 0 & 1 \\ -1 & c(2y-1)\end{vmatrix} = 1.$$

Hence the map is area-preserving.

(ii) The fixed points are at the roots of

$$y = x \quad\text{and}\quad y = -x + cy(y-1).$$

Substituting from the first of these in the second we obtain $cy^2 = y(c+2)$: thus $y=0$ and $y=(c+2)/c$. The fixed points are therefore

$$(0,0) \quad\text{and}\quad \left(\frac{c+2}{c},\frac{c+2}{c}\right).$$

The trace of the Jacobian matrix is $T=c(2y-1)$, and for the fixed point to be stable we need $|T|<2$. If $y=0$, $T=-c$, so this fixed point is stable if $|c|<2$. If $y=(c+2)/c$, $T=c+4$, so the condition for this fixed point to be stable is $-6<c<-2$.