(11)

b) i) $\xi_{n+1} = \begin{cases} 2\xi_n & 0 \le \xi_n \le \frac{1}{2} \\ 2(1-\xi_n) & 0 \le \xi_n \le 1 \end{cases}$

$\xi_0 = 4/7 > 1/2$
$\xi_{n+1} = 2(1-4/7) = 2\cdot 3/7 = 6/7 > 1/2$
$\xi_2 = 2(1-6/7) = 2\cdot 1/7 = 2/7 < 1/2$ ✓
$\therefore \xi_3 = 2\cdot 2/7 = 4/7 = \xi_0 \therefore 4/7$ has period 3

$\xi_0 = 8/15 > 1/2$
$\xi_1 = 2(1-8/15) = 2\cdot 7/15 = 14/15 > 1/2$
$\xi_2 = 2(1-14/15) = 2\cdot 1/15 = 2/15 < 1/2$
$\xi_3 = 2\cdot 2/15 = 4/15 < 1/2$
$\xi_4 = 2\cdot 4/15 = 8/15 = \xi_0 \therefore 8/15$ has period 4 ✓

$\xi_0 = 16/31 > 1/2$
$\xi_1 = 2(1-16/31) = 2\cdot 15/31 = 30/31 > 1/2$
$\xi_2 = 2(1-30/31) = 2\cdot 1/31 = 2/31 < 1/2$
$\xi_3 = 2\cdot 2/31 = 4/31 < 1/2$
$\xi_4 = 2\cdot 4/31 = 8/31 < 1/2$
$\xi_5 = 2\cdot 8/31 = 16/31 = \xi_0 \therefore 16/31$ has period 5 ✓

Generally a point of form $\frac{2^{k-1}}{2^k-1}$ has period k

Proof. $\xi_0 = \frac{2^{k-1}}{2^k-1}$ $\frac{1}{2}$

$\xi_1 = 2\left(1 - \frac{2^{k-1}}{2^k-1}\right) = 2\left(\frac{2^k-1-2^{k-1}}{2^k-1}\right)$
$= 2\left(\frac{2^{k-1}-1}{2^k-1}\right) = \frac{2^k-2}{2^k-1} > \frac{1}{2}$

$\xi_2 = 2\left(1 - \left(\frac{2^k-2}{2^k-1}\right)\right) = 2\left(\frac{2^k-1-2^k+2}{2^k-1}\right)$
$= 2\cdot\frac{1}{2^k-1} = \frac{2}{2^k-1}$

After $k-1$ iterations (none of the $\xi_n > 1/2$ during this operation) we have

$\xi_{k-1} = \frac{2^{k-2}}{2^k-1} < \frac{1}{2}$

$\xi_k = 2\cdot\frac{2^{k-2}}{2^k-1} = \frac{2^{k-1}}{2^k-1} = \xi_0$

$\therefore \xi = \frac{2^{k-1}}{2^k-1}$ has period k ✓

5/5