You must focus on a particular value of n.

$$H(\theta, I, t) = AI^{2/3} + F\left(\frac{2AI^{2/3}}{3a} + \sum_{n=1}^{\infty} \frac{-4AI^{2/3}}{a\pi^2 n^2}\cos n\theta\right)\cos\Omega t$$

$$= AI^{2/3} + \frac{2FAI^{2/3}}{3a}\cos\Omega t - \sum_{n=1}^{\infty}\frac{4FAI^{2/3}}{a\pi^2 n^2}\cos n\theta \cos\Omega t$$

$$\cos n\theta \cos\Omega t = \frac{1}{2}\cos(n\theta + \Omega t) + \frac{1}{2}\cos(n\theta - \Omega t)$$

$$\therefore H(\theta, I, t) = AI^{2/3} + \frac{2FAI^{2/3}}{3a}\cos\Omega t - \sum_{n=1}^{\infty}\frac{2FAI^{2/3}}{a\pi^2 n^2}(\cos(n\theta + \Omega t) + \cos(n\theta - \Omega t))$$

the $\cos\Omega t$ and $\cos(n\theta + \Omega t)$ vary quickly by comparison with $\cos(n\theta - \Omega t)$ so over a long period we can assume that the mean effect of these terms is zero. Thus for motion near the resonant action I_n we can write

$$H_n(\theta, I, t) = AI_n^{2/3} - \frac{2FAI_n^{2/3}}{a\pi^2 n^2}\cos(n\theta - \Omega t)$$

Apply the canonical transformation $(\theta, I) \to (\phi, J)$

$$I = J \qquad \phi = \theta - \frac{\Omega t}{n}$$

$$\phi = \frac{\partial F}{\partial J}(J, \phi) = \theta - \frac{\Omega t}{n} \Rightarrow F(J, \phi) = J\left(\theta - \frac{\Omega t}{n}\right)$$

$$H(\phi, J, t) = AJ^{2/3} - \frac{2FAJ^{2/3}}{a\pi^2 n^2}\cos n\phi + \frac{\partial F}{\partial t}$$

No. the point of the transformation is to remove the explicit

$$= AJ^{2/3} - \frac{\Omega J}{n} - \frac{2FAJ^{2/3}}{a\pi^2 n^2}\cos n\phi$$

Expand this Hamiltonian about the resonant action by putting $J = I_r + X$, where it is assumed X is small. Then $AJ^{2/3} = H_0$ so

$$H_0(I_r + X) - \frac{\Omega}{n}(I_r + X)$$

Use n rather than r as in text.

$$\approx H_0(I_r) + \frac{\partial H_0}{\partial I}(I_r)X - \frac{\Omega I_r}{n} - \frac{\Omega X}{n} + \frac{1}{2}\frac{\partial^2 H_0}{\partial I^2}(I_r)X^2$$

$$= \left(H_0(I_r) - \frac{\Omega I_r}{n}\right) + \left(\frac{\partial H}{\partial I}(I_r) - \frac{\Omega}{n}\right)X + \frac{1}{2}\frac{\partial^2 H_0}{\partial I^2}(I_r)X^2$$

$H_0(I_r) - \frac{\Omega}{n} I_r$ is constant so can be ignored.

$$\left(\frac{\partial H}{\partial I}(I_r) - \frac{\Omega}{n}\right)X = \left(\omega(I_r) - \frac{\Omega}{n}\right)X = 0$$