Question 4

Consider the motion of a particle of mass m moving in the potential

$$V(q) = \begin{cases} aq, & q \geq 0, \\ \infty, & q < 0, \end{cases}$$

where a is a positive constant.

(a) Show that all motion is bound and periodic, and that the action, I, the energy, E, and the frequency, ω, are connected by the relations

$$E(I) = AI^{2/3}, \quad \omega(I) = \tfrac{2}{3}AI^{-1/3},$$

where $A = \left(\dfrac{3\pi a}{2\sqrt{2m}}\right)^{2/3}$.

Show, further, that the angle variable θ, for $0 \leq \theta \leq \pi$, may be chosen to be

$$\theta = \pi\left(1 - \sqrt{1 - aq/E}\right),$$

so that $q(\theta, I) = I^{2/3}\dfrac{A}{a}\left(\dfrac{2\theta}{\pi} - \dfrac{\theta^2}{\pi^2}\right)$. [7]

(b) Show that the Fourier series for $q(\theta, I)$ is

$$q(\theta, I) = Q_0(I) + \sum_{n=1}^{\infty} Q_n(I)\cos n\theta,$$

where $Q_0 = \dfrac{2A}{3a}I^{2/3}$, $Q_n = -\dfrac{4A}{a\pi^2 n^2}I^{2/3}$. [6]

[Hint: you may find the following integrals useful:

$$\int_0^{\pi} d\theta\, \theta \cos n\theta = \frac{1}{n^2}(\cos n\pi - 1), \quad n \geq 1,$$

$$\int_0^{\pi} d\theta\, \theta^2 \cos n\theta = \frac{2\pi}{n^2}\cos n\pi, \quad n \geq 1.]$$

(c) The system is perturbed by a periodic force, so that the Hamiltonian becomes

$$H(q, p, t) = \frac{p^2}{2m} + V(q) + Fq\cos\Omega t.$$

Show that the nth resonance is at the action

$$I_n = \frac{\pi^2 a^2}{3m}\left(\frac{n}{\Omega}\right)^3,$$

and that near here the Hamiltonian can be cast into the form

$$K(\phi, X) = -\frac{A}{9I_n^{4/3}}X^2 - \frac{2AFI_n^{2/3}}{a\pi^2 n^2}\cos n\phi.$$

Hence, or otherwise, deduce that the separatrix width $S^{(n)}$ of the nth resonance island is given by

$$S^{(n)} = \frac{4a\pi n^2\sqrt{aF}}{m\Omega^3}.$$ [8]

(d) Using the result of *Unit 11*, Section 11.5, that for resonance islands not to intersect we must have $\Delta I_n > \frac{1}{2}(S^{(n)} + S^{(n+1)})$, deduce that the resonant approximation is valid near the nth resonance only if

$$F < \frac{a\pi^2}{16}.$$ [4]