(b) This part of the question is concerned with the iteration under the tent map, defined in *Unit 12*, Subsection 12.3.2.

(i) Show that $\frac{4}{7}$ is a point of period 3 for the tent map, $\frac{8}{15}$ is a point of period 4, and $\frac{16}{31}$ is a point of period 5.

For any integer $k \geq 3$, find a point of period k for the tent map. [5]

(ii) Consider the number ξ_0 whose binary representation is

$$0.100\,1000\,10000\,100000\ldots$$

In words, the binary representation of ξ_0 is a succession of groups of digits comprising a 1 followed by several 0s; the number of zeros increases by one with each successive group; initially, there are two 0s.

Show that no iterate of ξ_0 under the tent map can begin with either $0.011\ldots$ or $0.101\ldots$. Deduce that the orbit of ξ_0 is neither periodic nor dense. [5]

(c) Consider the second-order discrete system

$$x_{n+1} = y_n,$$
$$y_{n+1} = -x_n + cy_n(y_n - 1),$$

where c is a parameter.

(i) Show that this map is area-preserving. [3]

(ii) Locate the fixed points, and for each fixed point find the range of values of c for which it is stable. [6]

Question 3

Suppose that the Hamiltonian

$$H_0(q,p) = \tfrac{1}{2}p^2 + V(q)$$

has a hyperbolic fixed point at $q = q_h$, $p = 0$, such that one branch of the separatrix forms a closed loop, as illustrated in *Unit 13*, Figure 13.4. Thus any solution of the equations of motion on the separatrix, $(q_s(t), p_s(t))$, satisfies

$$\lim_{t\to\pm\infty} q_s(t) = q_h, \qquad \lim_{t\to\pm\infty} p_s(t) = 0.$$

(a) Show that, by choosing a suitable point as initial conditions for the solution of the equations of motion on the separatrix, it may be assumed that the function $q_s(t)$ is even and the function $p_s(t)$ is odd; that is, that

$$q_s(-t) = q_s(t) \quad \text{and} \quad p_s(-t) = -p_s(t),$$

respectively. [*10*]

Consider the periodically forced system with Hamiltonian

$$H(q,p,t) = H_0(q,p) + \varepsilon q \sin \Omega t.$$

(b) Use part (a) and a result proved in *Unit 13* to deduce that, for this system, motion near the unperturbed separatrix is necessarily chaotic, provided that $|\varepsilon|$ is sufficiently small. [*15*]