Question 2

(a) For the quadratic first-order discrete system

$$x_{n+1} = x_n(x_n + \lambda), \qquad \lambda > 0,$$

find the value of the parameter λ at which period doubling of a period 2 to a period 4 orbit occurs. [4]

(b) This part of the question is concerned with iteration under the tent map, defined in *Unit 12* Subsection 12.3.2.

(i) Show that, for the tent map, $\frac{4}{7}$ is a point of period 3, $\frac{8}{15}$ is a point of period 4, and $\frac{16}{31}$ is a point of period 5.

For any integer $k \geq 3$, find a point of period k for the tent map. [4]

(ii) Consider the number ξ_0 whose binary representation is

$$0.100\,1000\,10000\,100000\ldots$$

In words, the binary representation of ξ_0 is a succession of groups of digits comprising a 1 followed by several 0s; the number of zeros increases by one with each successive group; initially there are two 0s.

Show that no iterate of ξ_0 under the tent map can begin with either $0.011\ldots$ or $0.101\ldots$. Deduce that the orbit of ξ_0 is neither periodic nor dense. [4]

(c) Consider the second-order discrete system

$$\mathbf{M} : \begin{cases} y_{n+1} = y_n + k\cos x_n, \\ x_{n+1} = x_n + y_{n+1}, \end{cases}$$

where k is a positive parameter.

(i) Show that this map is area-preserving. [1]

(ii) Show that $\mathbf{M}$ has a fixed point f_1 at $(\pi/2, 0)$, and find the other fixed points in the interval $0 \leq x \leq 2\pi$. [2]

(iii) Show that f_1 is stable for $0 \leq k < 4$ and unstable for $k > 4$. [2]

(iv) Show that $\mathbf{M}^2$ has fixed points at the roots of

(a) $2\pi - 4x = k\cos x, \quad y = 2x - \pi,$

(b) $k\cos x = -2\pi, \quad y = \pi.$ [4]

(v) Sketch the graphs of the curves $w(x) = 2\pi - 4x$ and $w(x) = k\cos x$ for x near $\pi/2$ and k near 4, and discuss how the nature of the fixed points defined by condition (a) changes as k increases through 4. [4]

Question 3

Suppose that the Hamiltonian

$$H_0(q,p) = \tfrac{1}{2}p^2 + V(q)$$

has a hyperbolic fixed point at $q = q_h$, $p = 0$, such that one branch of the separatrix forms a closed loop, as illustrated in *Unit 13* Figure 13.4. Thus any solution of the equations of motion on the separatrix, $(q_s(t), p_s(t))$, satisfies

$$\lim_{t\to\pm\infty} q_s(t) = q_h, \qquad \lim_{t\to\pm\infty} p_s(t) = 0.$$

(i) Show that, by choosing a suitable point as initial conditions for the solution of the equations of motion on the separatrix, it may be assumed that the function $q_s(t)$ is even and the function $p_s(t)$ is odd; that is,

$$q_s(-t) = q_s(t) \qquad \text{and} \qquad p_s(-t) = -p_s(t)$$

respectively. [10]