Consider the periodically forced system with Hamiltonian

$$H(q,p,t) = H_0(q,p) + \varepsilon q \sin \Omega t.$$

(ii) Use part (i) and a result proved in *Unit 13* to deduce that, for this system, motion near the unperturbed separatrix is necessarily chaotic, provided that $|\varepsilon|$ is sufficiently small. [15]

Question 4

Consider the motion of a particle of mass m moving in the potential

$$V(q) = \begin{cases} 0, & |q| \le L, \\ \infty, & |q| > L, \end{cases}$$

where L is a positive constant.

(i) Show that all motion is bound and periodic, and that the action, I, the energy, E, and the frequency, ω, are connected by the relations

$$E(I) = \tfrac{1}{2}AI^2, \qquad \omega(I) = AI,$$

where $A = \dfrac{\pi^2}{4mL^2}$.

Show, further, that the angle variable, θ, may be chosen so that for $-\pi/2 \le \theta \le \pi/2$,

$$q = 2L\theta/\pi \qquad \text{and} \qquad q(\theta) = q(\pi - \theta).$$ [7]

(ii) Show that the Fourier series for $q(\theta, I)$ is

$$q(\theta, I) = \sum_{n=1}^{\infty} Q_n(I) \sin(2n+1)\theta,$$

where $Q_n = \dfrac{8L}{\pi^2}\dfrac{(-1)^n}{(2n+1)^2}$. [6]

[Hint: You may find the following integral useful:

$$\int_0^{\pi/2} d\theta\, \theta \sin(2n+1)\theta = \frac{1}{(2n+1)^2}\cos n\pi.]$$

(iii) The system is perturbed by a periodic force, so that the Hamiltonian becomes

$$H(q,p,t) = \frac{p^2}{2m} + V(q) + Fq\sin\Omega t.$$

Show that the nth resonance is at the action

$$I_n = \frac{4m\Omega L^2}{\pi^2(2n+1)},$$

and that near here the Hamiltonian can be cast into the form

$$K(\phi, J) = \tfrac{1}{2}AJ^2 + \frac{4LF}{\pi^2(2n+1)^2}\sin(2n+1)\phi.$$

Hence, or otherwise, deduce that the separatrix width $S^{(n)}$ of the nth resonance island is given by

$$S^{(n)} = \frac{16\sqrt{mFL^3}}{(2n+1)\pi^2}.$$ [8]

(iv) Using the result of *Unit 11* Section 11.5, that for resonance islands not to intersect we must have $\Delta I_n > \frac{1}{2}(S^{(n)} + S^{(n+1)})$, deduce that for large n the resonant approximation is valid near the nth resonance only if

$$F < \frac{mL\Omega^2}{16n^2}.$$ [4]