

(Covering Units 11, 12 and 13 of Block IV.)

Each question is marked out of 25. Your overall grade will be determined by the sum of your marks for all four questions.

Question 1

Consider the periodic time-dependent Hamiltonian

$$H(q, p, t) = \frac{1}{2}(p^2 + q^2) + \frac{1}{4}\epsilon q^2 p^2 + \frac{1}{2}\epsilon q^2 \sin \Omega t,$$

where ϵ is a small parameter such that $0 \leq \epsilon \ll 1$.

- (i) Show that, when it is expressed in terms of the angle-action variables (θ, I) for the unperturbed Hamiltonian ($\epsilon = 0$), this Hamiltonian can be written in the form

$$H(\theta, I, t) = I + \frac{1}{8}\epsilon I^2 + \frac{1}{4}\epsilon I \sin(2\theta - \Omega t) + \frac{1}{8}\epsilon I \{4 \sin \Omega t - I \cos 4\theta - 2 \sin(2\theta + \Omega t)\}.$$

Deduce that there is a resonance at $\Omega = 2$.

[5]

- (ii) Find a time-dependent canonical transformation to new conjugate variables (ϕ, J) such that the Hamiltonian $K(\phi, J, t)$ of the system in terms of the new variables is given by

$$K(\phi, J, t) = (1 - \frac{1}{2}\Omega) J + \frac{1}{8}\epsilon J^2 + \frac{1}{4}\epsilon J \sin 2\phi + \frac{1}{8}\epsilon J \{4 \sin \Omega t - J \cos(4\phi + 2\Omega t) - 2 \sin 2(\phi + \Omega t)\}.$$

Show that for $\Omega \simeq 2$, the slowly varying part of this Hamiltonian can be written in the form

$$\bar{K}(\phi, J) = \frac{1}{4}\epsilon \{(\nu + \sin 2\phi)J + \frac{1}{2}J^2\}, \quad J \geq 0,$$

where $\frac{1}{4}\epsilon\nu = 1 - \frac{1}{2}\Omega$.

[10]

- (iii) Show that if $-1 < \nu < 1$, the mean motion system (the system with Hamiltonian $\bar{K}(\phi, J)$) has a stable fixed point at

$$4\phi = -\pi, \quad J = 1 - \nu.$$

[6]

- (iv) Write down an approximation to the stable periodic orbit of the original Hamiltonian in the original (q, p) -representation corresponding to this fixed point.

[4]

Question 2

- (a) Consider the first-order discrete system

$$x_{n+1} = F_\lambda(x_n), \quad \text{where } F_\lambda(x) = \lambda \sin(\pi x).$$

Show that, for $0 < \lambda \leq 1$,

- (i) $F_\lambda(x)$ maps the interval $[0, 1]$ into itself; [1]
 (ii) the origin is a fixed point for the system, while for $\pi\lambda > 1$ there is a second fixed point in the interval $[0, 1]$; [2]
 (iii) for $\pi\lambda > 1$ the fixed point at the origin is unstable; [1]
 (iv) the value of λ at which the non-zero fixed point becomes unstable, and period doubling occurs, satisfies the equation

$$\pi\lambda \cos \sqrt{\pi^2\lambda^2 - 1} = -1.$$

[2]