MARK SCHEME FOR TMA MS323 03

Cut-off date 4th Augu

TMA MS323 03 Question 1: Specimen Solution

(a) The graph of the functions $y(x) = \cos 2x$ and $y(x) = 0.1/x$ are shown in the figure.

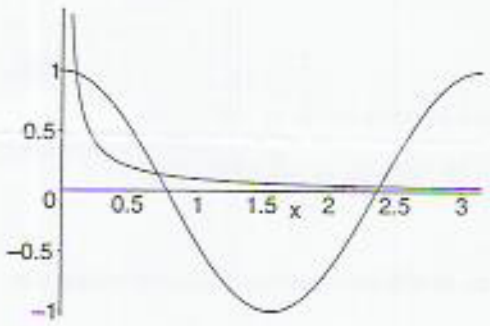

Figure 1 Graphs of the functions $y(x) = \cos 2x$ and $y = 0.1/x$.

From these graphs we see that for any positive ε the equation $\cos 2x = \varepsilon/x$ has positive roots and that for small values of ε there are roots near $(2n+1)\pi/4$, $n = 0, 1, \cdots$.

In order to find the root near $\pi/4$ put $x = \pi/4 - w$ to give the equation

$$\sin 2w = \frac{2\delta}{1 - 4w/\pi}, \quad \text{with} \quad \delta = \frac{2\varepsilon}{\pi}$$

Clearly we require the smallest positive root of this equation. Since $|\delta| \ll 1$, $w = O(\delta)$ and a first approximation is

$$\sin 2w = 2\delta + O(\delta^2) \quad \text{giving} \quad w = \delta + O(\delta^2).$$

Thus we set

$$w = \delta + w_2\delta^2 + w_3\delta^3 + O(\delta^4).$$

On working to $O(\delta^3)$ we have, on using the expansions given in the Handbook, Section 1.6,

$$\begin{aligned}\sin 2w &= 2w - \tfrac{4}{3}w^3 + O(w^5)\\ &= 2\delta + 2w_2\delta^2 + \left(2w_3 - \tfrac{4}{3}\right)\delta^3 + O(\delta^4),\end{aligned}$$

and

$$\begin{aligned}\frac{2\delta}{1 - 4w/\pi} &= 2\delta\left(1 + \frac{4w}{\pi} + \frac{16w^2}{\pi^2} + O(w^3)\right)\\ &= 2\delta + \frac{8}{\pi}\delta^2 + \left(\frac{32}{\pi^2} + \frac{8w_2}{\pi}\right)\delta^3 + O(\delta^4)\end{aligned}$$

Thus the original equation can be written as

$$2\delta + 2w_2\delta^2 + \left(2w_3 - \tfrac{4}{3}\right)\delta^3 = 2\delta + \frac{8}{\pi}\delta^2 + \left(\frac{32}{\pi^2} + \frac{8w_2}{\pi}\right)\delta^3 + O(\delta^4)$$

Hence, on equating the coefficients of δ^2 and δ^3 we find

$$w_2 = \frac{4}{\pi} \quad \text{and} \quad 2w_3 - \frac{4}{3} = \frac{32}{\pi^2} + \frac{8w_2}{\pi} \quad \text{or} \quad w_3 = \frac{2}{3} + \frac{32}{\pi^2},$$