TMA MS323 03 Question 2: Specimen Solution

(a) Consider the system with the Hamiltonian

$$H = H_0(q,p) + \lambda V_1(q), \quad H_0 = \frac{1}{2m}p^2 + V(q), \quad 0 \le \lambda \le 1,$$

and let (θ, I) be the angle-action variables when $\lambda = 0$ and (ϕ, J) these variables when $\lambda \neq 1$. Since $\lambda|V_1| \ll |V|$ the Hamiltonian, $K(J)$, in the (ϕ, J)-representation is given approximately by first-order perturbation theory, as described in *Unit 8*, page 24,

$$K(J) = H_0(J) + \lambda\overline{V}(J) \quad \text{where} \quad \overline{V}(J) = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, V_1(q(\phi, J)) = \frac{1}{T}\int_0^T dt\, V_1(q(t)),$$

where $q(t)$ is the unperturbed motion due to the Hamiltonian H_0 with period T.

If we now allow λ to vary slowly, so that it changes relatively little during an unperturbed period

$$\frac{\lambda(t+T) - \lambda(t)}{\lambda} \ll 1 \quad \text{or} \quad \frac{T\dot{\lambda}}{\lambda} \ll 1,$$

then the action J is an adiabatic invariant, *Unit 9* Section 9.5, and remains constant as λ changes from 0 to 1. Therefore, if the initial energy is $E_i = H_0(J)$ and the final energy is $E_f = H_0(J) + \overline{V}(J)$ we have

$$\Delta E = E_f - E_i = \overline{V}(J) = \frac{1}{T}\int_0^T dt\, V_1(q(t)).$$

(b-i) This question is about the theory described in *Unit 9*, Section 9.3. The basic equation of motion is Equation 9.14, page 61, with $D(x)$ replaced by $y(x)$, with $C = v\cos\beta$ and with energy $E = mv^2/2$. Thus Equation 9.14 becomes

$$\dot{x} = \pm v\sqrt{1 - \cos^2\beta\left(\frac{1+b|x|}{1+a|x|}\right)^2}. \tag{1}$$

The turning points are at the points where $\dot{x} = 0$, that is, for $x > 0$,

$$\frac{1+ax}{1+bx} = \cos\beta. \tag{2}$$

Now the function on the left of this equation monotonically decreases from 1, at $x = 0$, to $\frac{a}{b}$ at $x = \infty$. Since $\cos\beta \le 1$ there is a root, and hence a turning point, only if

$$\cos\beta > \frac{a}{b}.$$

By rearranging equation (2) we see that the positive turning point is at

$$x = A = \frac{1 - \cos\beta}{b\cos\beta - a}$$

which is also the amplitude of the motion. The negative turning point is at $x = -A$.

(b-ii) From Equation (1) we see that $\dot{x}$ is an even function of x, so the period is just four times the time taken to go from $x = 0$ to $x = A$, that is

$$T = 4\int_0^A \frac{dx}{\dot{x}} = \frac{4}{v}\int_0^A dx\left(1 - \cos^2\beta\left(\frac{1+b|x|}{1+a|x|}\right)^2\right)^{-1/2}$$

$$= \frac{4}{v}\int_0^A dx\,\frac{1+ax}{\sqrt{(1+ax)^2 - (1+bx)^2\cos^2\beta}}.$$

Now use equation (2) to write $\cos\beta$ in terms of A and factorise the term inside the square root to give

$$T = (1+Ab)\frac{4}{v}\int_0^A dx\,\frac{1+ax}{\sqrt{(1+Ab)^2(1+ax)^2 - (1+Aa)^2(1+bx)^2}}$$

$$= \frac{4(1+Ab)}{v\sqrt{b-a}}\int_0^A dx\,\frac{1+ax}{\sqrt{A-x}}\,\frac{1}{\sqrt{2 + A(a+b) + (a+b+2Aab)x}}.$$