giving

$$x = \frac{\pi}{2} - w = \frac{\pi}{2} - \delta - \frac{4\delta^2}{\pi} - \left(\frac{2}{3} + \frac{32}{\pi^2}\right)\delta^3 + O(\delta^4), \quad \text{with} \quad \delta = \frac{2\varepsilon}{\pi}.$$

(b-i) Put

$$x(t) = x_0(t) + \varepsilon x_1(t) + O(\varepsilon^2)$$

with $x_0(0) = A$ and $x_1(0) = 0$. The differential equation becomes

$$\begin{aligned}\frac{dx_0}{dt} + \varepsilon\frac{dx_1}{dt} &= e^{2x_0+2\varepsilon x_1} + \varepsilon x_0^2 + O(\varepsilon^2)\\ &= e^{2x_0}(1 + 2\varepsilon x_1) + \varepsilon x_0^2 + O(\varepsilon^2).\end{aligned}$$

On equating the coefficients of the powers of ε the following two equations are obtained:

$$\begin{aligned}&\frac{dx_0}{dt} = e^{2x_0}, \quad x_0(0) = A\\ &\frac{dx_1}{dt} = 2x_1e^{2x_0} + x_0^2, \quad x_1(0) = 0.\end{aligned} \tag{1}$$

The solution of the first of these is obtained by separation of variables,

$$\int_A^{x_0} dx\, e^{-2x} = \int_0^t dt \quad \text{or} \quad e^{-2x_0} = B - 2t \quad \text{or} \quad x_0 = -\tfrac{1}{2}\ln(B - 2t),$$

where $B = e^{-2A}$.

The equation for x_1 is best written using x_0 as the independent variable: using the chain rule,

$$\frac{dx_1}{dt} = \frac{dx_1}{dx_0}\frac{dx_0}{dt} = \frac{dx_1}{dx_0}e^{2x_0}$$

so equation (1) becomes

$$\frac{dx_1}{dx_0} - 2x_1 = x_0^2e^{-2x_0} \quad \text{or} \quad \frac{d}{dx_0}\left(x_1e^{-2x_0}\right) = x_0^2e^{-4x_0}.$$

This last equation can be integrated directly, and since $x_1 = 0$ when $x_0 = A$ we have

$$x_1e^{-2x_0} = \int_A^{x_0} dx\, x^2e^{-4x} = \frac{1}{32}B^2(1 + 4A + 8A^2) - \frac{1}{32}e^{-4x_0}(1 + 4x_0 + 8x_0^2),$$

since

$$\int dx\, x^2e^{-4x} = -\frac{1}{32}e^{-4x}(1 + 4x + 8x^2).$$

On writing x_0 as a function of t and rearranging the above expression we obtain

$$x_1(t) = \frac{B^2(1 + 4A + 8A^2)}{32(B - 2t)} - \frac{1}{32}\left(1 - 2\ln(B - 2t) + 2\ln(B - 2t)^2\right)(B - 2t).$$

Hence the approximate solution is

$$x(t) = -\tfrac{1}{2}\ln(B - 2t) + \varepsilon\frac{B^2(1 + 4A + 8A^2)}{32(B - 2t)} - \frac{\varepsilon}{32}\left(1 - 2\ln(B - 2t) + 2\ln(B - 2t)^2\right)(B - 2t).$$

(b-ii) The solution of the unperturbed equation, $\dot{x} = e^{2x}$, terminates when $t = B = e^{-2A}$, that is $x(t) \to \infty$ as $t \to B$.

Since $|\varepsilon x^2| \ll e^x$ for large values of x we expect the solution of the perturbed equation to behave similarly. Suppose that the exact solution terminates at $t = C$, then

$$C = \int_A^{\infty} dx\, \frac{e^{-2x}}{1 + \varepsilon x^2e^{-2x}}.$$

This integral exists, so the motion terminates at a finite time.