(Covering *Units 8, 9* and *10* of Block III.)

Each question is marked out of 25. Your overall grade will be determined by the sum of your marks for all four questions.

Question 1

(a) By sketching graphs of the functions $y = \cos 2x$ and $y = \varepsilon/x$, show that the equation

$$\cos 2x = \frac{\varepsilon}{x}, \quad |\varepsilon| \ll 1,$$

has a root near $x = \pi/4$.

Use perturbation theory to show that an approximation to this root is

$$x = \frac{\pi}{4} - \delta - \frac{4}{\pi}\delta^2 - \frac{2(48+\pi^2)}{3\pi^2}\delta^2 + O(\delta^4), \qquad \text{where} \quad \delta = 2\varepsilon/\pi.$$ [8]

(b) (i) Show that the differential equation

$$\frac{dx}{dt} = e^{2x} + \varepsilon x^2, \quad x(0) = A > 0,$$

where ε is a small parameter, has the approximate solution

$$x(t) = -\frac{1}{2}\ln(B-2t) + \varepsilon\frac{B^2(1+4A+8A^2)}{32(B-2t)} - \frac{\varepsilon}{32}\left(1 - 2\ln(B-2t) + 2\ln(B-2t)^2\right)(B-2t),$$

where $B = \exp(-2A)$ and $0 \le t < B/2$. [12]

(ii) Why is the variation of t restricted to a finite interval? [5]

Question 2

(a) Suppose that the potentials $V(q)$ and $V(q) + V_1(q)$ both support librational motion, with the property that the two sets of phase curves can be continuously deformed into each other, and that $|V_1| \ll |V|$ for all q. By considering the system with the potential $V(q) + \lambda V_1(q)$, show that the energy of the system in which the potential is changed slowly from $V(q)$ to $V(q) + V_1(q)$ changes, approximately, by

$$\Delta E = \frac{1}{T}\int_0^T dt\, V_1(q(t)),$$

where $q(t)$ is the initial librational motion with period T. [5]

(b) Consider the motion of a particle of mass m elastically bouncing in the (x, y)-plane between the horizontal line $y = 0$ and the curve

$$y = \frac{1 + a|x|}{1 + b|x|} \quad \text{with} \quad 0 < a < b \ll 1.$$

(i) If the particle starts at the origin with speed v and with the angle between the initial velocity and the y-axis denoted by β, show that the motion is bound provided that

$$\cos\beta > \frac{a}{b}.$$

Show, further, that the amplitude A of the motion is given approximately by

$$A = \frac{1 - \cos\beta}{b\cos\beta - a}.$$ [5]