(ii) Show that the period of the bound motion can be written in the form of the integral

$$T = \frac{4(1+Ab)}{v\sqrt{b-a}} \int_0^A dx \, \frac{1+ax}{\sqrt{A-x}} \frac{1}{\sqrt{2+(a+b)(A+x)+2Aabx}}. \qquad [7]$$

(iii) By making the substitution $x = A\sin^2\theta$ and assuming both aA and bA to be small, show that

$$T = \frac{8(1+Ab)}{v\sqrt{b-a}} \sqrt{\frac{A}{2}} \left\{ 1 + \frac{A}{12}(3a-5b) + \cdots \right\}. \qquad [8]$$

Question 3

Provide a qualitative description of the mean motion of a particle of mass m moving according to the Hamiltonian

$$H(q,p,t) = \frac{p^2}{2m} + F\sin(\Omega t + \delta)\tan h^2(q/a),$$

where F, δ and a are constants, and the frequency Ω is large. In particular, show that a particle starting at $q \ll -a$ with velocity $\dot{q} = v_i > 0$ tends asymptotically to $q = \infty$ only if v_i is above a certain critical velocity, v_c. Find an expression for this critical velocity in terms of F, a, m and Ω. [25]

Question 4

Consider the equation

$$\frac{dx}{dt} = -\lambda x + \varepsilon x(t-a), \quad x(0) = A,$$

where A, a and λ are positive constants, and ε is a small parameter. Note that $x(t-a)$ means the function $x(t)$ evaluated at $t-a$.

(a) By assuming that the solution may be expressed as a power series in ε,

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots,$$

show that

$$x_0(t) = Ae^{-\lambda t},$$

and that the $x_k(t)$ satisfy the equations

$$\dot{x}_k + \lambda x_k = x_{k-1}(t-a),$$

and hence that

$$x_k(t) = e^{-\lambda t} \int_0^t du \, e^{\lambda u} x_{k-1}(u-a). \qquad [10]$$

(b) By solving the first two of these equations, show that an approximate solution of the original equation is

$$x(t) = Ae^{-\lambda t} + A\varepsilon t e^{-\lambda(t-a)} + \tfrac{1}{2}A\varepsilon^2 t(t-2a)e^{-\lambda(t-2a)} + \cdots. \qquad [15]$$