Question 1 below, on *Unit 1*, forms the first part of Tutor-marked Assignment MS323 01. The remainder of the TMA (on *Units 2, 3* and *4*) can be found immediately following Question 1 in this booklet.

Question 1 is marked out of 25. (The whole TMA is marked out of 100.)

Please send your answers to Question 1 to your tutor, along with an assignment form (PT3). You will find instructions on how to fill in the PT3 form in the *Student Handbook*: be sure to fill in the Assignment Number on this form as

MS323 01.

Your tutor will mark and comment on your solution to Question 1, and will send your script back to you directly to give you some early feedback on the course. He or she will retain your PT3 to enter your marks for the rest of this assignment on it. The form will then be sent to you via Walton Hall, so that your mark can be recorded.

Question 1

(a) Compute the Jacobian matrix of each of the following transformations.

(i) $u = e^{kx}$, $v = \alpha xy$.

(ii) $u = 2e^{k}\sqrt{x}\cos\alpha y$, $v = e^{-k}\sqrt{x}\sin\alpha y$, where k and α are constant and $x > 0$.

Determine whether there are any values of k and α for which either transformation is area-preserving. [*4*]

(b) (i) Sketch the graphs of the functions $y = \cos x$ and $y = e^{-x}$, and hence show that the equation

$$\cos x = \varepsilon e^{-x}, \qquad 0 < \varepsilon \ll 1,$$

has a root $x(\varepsilon)$ such that $x(0) = \pi/2$.

By differentiating the above equation, obtain expressions for $d^k x/d\varepsilon^k$, $k = 1$, 2 and 3. By evaluating these at $x = \pi/2$ and $\varepsilon = 0$, show that the third-order Taylor series for $x(\varepsilon)$ is

$$x(\varepsilon) = \tfrac{1}{2}\pi - \varepsilon e^{-\pi/2} - \varepsilon^2 e^{-\pi} - \tfrac{5}{3}\varepsilon^3 e^{-3\pi/2} + O(\varepsilon^4).$$ [*10*]

(ii) Consider the function of two variables

$$f(x, y) = \cos x - ye^{-x}.$$

Show that the second-order Taylor expansion about the point $(\pi/2, 0)$ is

$$f(x, y) = -\left(x - \frac{\pi}{2}\right) - ye^{-\pi/2} + y\left(x - \frac{\pi}{2}\right)e^{-\pi/2} + \cdots.$$ [*6*]

Using these results, show that an approximate solution, $x(y)$, to the equation $f(x, y) = 0$ such that $x(0) = \pi/2$ is

$$x \simeq \frac{\pi}{2} - \frac{ye^{-\pi/2}}{1 - ye^{-\pi/2}}.$$

Show that this agrees with the result obtained in part (b)(i) to $O(y^2)$. [*5*]