Question 1 below, on *Unit 1*, forms the first part of Tutor-marked Assignment MS323 01. The remainder of the TMA (on *Units 2, 3* and *4*) can be found immediately following Question 1 in this booklet.

Question 1 is marked out of 25. (The whole TMA is marked out of 100.)

Please send your answers to Question 1 to your tutor, along with an assignment form (PT3). You will find instructions on how to fill in the PT3 form in the *Student Handbook*: be sure to fill in the Assignment Number on this form as

MS323 01.

Your tutor will mark and comment on your solution to Question 1, and will send your script back to you directly to give you some early feedback on the course. He or she will retain your PT3 to enter your marks for the rest of this assignment on it. The form will then be sent to you via Walton Hall, so that your mark can be recorded.

Question 1

(a) Compute the Jacobian matrix of each of the following transformations, and determine which is area-preserving.

(i) $u = \frac{1}{2}x^2, \quad v = y^2/x.$

(ii) $u = \tan x, \quad v = (y - k)\cos^2 x,$ where k is a constant. [*3*]

(b) (i) Write down the second-order Taylor expansion about $x = y = 0$ for the function

$$f(x, y) = \frac{y}{(1+x)^n} - x,$$

where n is an integer. [*3*]

Use this expansion to show that the equation

$$\frac{\alpha}{(1+x)^n} = x, \qquad 0 < \alpha \ll 1,$$

has the approximate solution $x = \alpha + O(\alpha^2)$. [*3*]

(ii) Using the above equation to define the function $x(\alpha)$, use the chain rule to show that this satisfies the differential equation

$$(1 + (n+1)x)\frac{dx}{d\alpha} = \frac{1}{(1+x)^{n-1}}.$$ [*5*]

By repeated differentiation, show that

$$(1 + (n+1)x)\frac{d^2x}{d\alpha^2} + (n+1)\left(\frac{dx}{d\alpha}\right)^2 = -\frac{n-1}{(1+x)^n}\frac{dx}{d\alpha},$$

$$(1 + (n+1)x)\frac{d^3x}{d\alpha^3} + 3(n+1)\frac{dx}{d\alpha}\frac{d^2x}{d\alpha^2} = \frac{n(n-1)}{(1+x)^{n+1}}\left(\frac{dx}{d\alpha}\right)^2 - \frac{n-1}{(1+x)^n}\frac{d^2x}{d\alpha^2}.$$ [*6*]

(iii) Using these results, show that

$$x''(0) = -2n \qquad \text{and} \qquad x'''(0) = 3n(3n+1),$$

and, using the Taylor series for $x(\alpha)$, show that

$$x(\alpha) = \alpha - n\alpha^2 + \tfrac{1}{2}n(3n+1)\alpha^3 + O(\alpha^4).$$ [*5*]