

(Covering Units 8, 9 and 10 of Block III.)

Each question is marked out of 25. Your overall grade will be determined by the sum of your marks for all four questions.

Question 1

- (a) By sketching graphs of the functions $y = \sin(x^2)$ and $y = \varepsilon/x$, show that the equation

$$\sin(x^2) = \frac{\varepsilon}{x}, \quad 0 < \varepsilon \ll 1, \quad x > 0$$

has roots near $x = \sqrt{n\pi}$ for $n = 1, 2, \dots$

Use perturbation theory to show that an approximation to this root is

$$x = \sqrt{n\pi} + \frac{(-1)^n \varepsilon}{2n\pi} - \frac{3\varepsilon^2}{8(n\pi)^{5/2}} + O(\varepsilon^3). \quad [10]$$

- (b) Show that the differential equation

$$\frac{dx}{dt} = \frac{1}{x} + \frac{\varepsilon}{x^2}, \quad x(0) = A > 0,$$

where ε is a small parameter, has the approximate solution

$$x(t) = \sqrt{2t + A^2} + \varepsilon \left(1 - \frac{A}{\sqrt{2t + A^2}} \right). \quad [15]$$

Question 2

- (a) Consider the Hamiltonian

$$H(q, p, \alpha) = H_0(q, p) - \alpha q,$$

where $H_0(q, p)$ is independent of α , and suppose that the Hamiltonian $H(q, p, \alpha)$ supports librational motion for values of α in some interval. Let the angle-action variables of this librational motion be (θ, I) , and let the Hamiltonian in this representation be $H(I, \alpha)$.

Consider the system with α replaced by $\alpha + \delta\alpha$, where $\delta\alpha$ is small, and use first-order perturbation theory to find an expression for the energy $E(I, \alpha + \delta\alpha)$ in terms of $E(I, \alpha)$ and $\delta\alpha$. By taking the limit as $\delta\alpha \rightarrow 0$, show that

$$\frac{\partial E}{\partial \alpha} = -\bar{q}(I, \alpha), \quad (1)$$

where $\bar{q}(I, \alpha)$ is the mean value of q in the motion with action I ,

$$\bar{q}(I, \alpha) = \frac{1}{2\pi} \int_0^{2\pi} d\theta q(\theta, I, \alpha). \quad [10]$$