

(Covering Units 8, 9 and 10 of Block III.)

Each question is marked out of 25. Your overall grade will be determined by the sum of your marks for all four questions.

Question 1

- (a) By sketching graphs of the functions
- $y = \cos x$
- and
- $y = \varepsilon/x$
- , show that the equation

$$\cos x = \frac{\varepsilon}{x}, \quad |\varepsilon| \ll 1,$$

has a root near $x = \pi/2$.

Use perturbation theory to show that an approximation to this root is

$$x = \frac{\pi}{2} - \delta - \frac{2}{\pi}\delta^2 - \left(\frac{1}{6} + \frac{8}{\pi^2}\right)\delta^3 + O(\delta^4), \quad \text{where } \delta = 2\varepsilon/\pi. \quad [8]$$

- (b) (i) Show that the differential equation

$$\frac{dx}{dt} = e^x + \varepsilon x^2, \quad x(0) = A > 0,$$

where ε is a small parameter, has the approximate solution

$$x(t) = -\ln(B-t) + \varepsilon \frac{B^2(1+2A+2A^2)}{4(B-t)} - \frac{\varepsilon}{4}(1-2\ln(B-t)+2\ln(B-t)^2)(B-t),$$

where $B = \exp(-A)$ and $0 \leq t < B$. [12]

- (ii) Why is the variation of
- t
- restricted to a finite interval?
- [5]

Question 2

- (a) Suppose that the potentials
- $V(q)$
- and
- $V(q) + V_1(q)$
- both support librational motion, with the property that the two sets of phase curves can be continuously deformed into each other, and that
- $|V_1| \ll |V|$
- for all
- q
- . By considering the system with the potential
- $V(q) + \lambda V_1(q)$
- , show that the energy of the system in which the potential is changed slowly from
- $V(q)$
- to
- $V(q) + V_1(q)$
- changes, approximately, by

$$\Delta E = \frac{1}{T} \int_0^T dt V_1(q(t)),$$

where $q(t)$ is the initial librational motion with period T . [5]

- (b) Consider the motion of a particle of mass
- m
- bouncing elastically in the
- (x, y)
- plane between the horizontal line
- $y = 0$
- and the curve

$$y = \frac{1+a|x|}{1+b|x|} \quad \text{with } 0 < a < b \ll 1.$$

- (i) If the particle starts at the origin with speed
- v
- and with the angle between the initial velocity and the
- y
- axis denoted by
- β
- , show that the motion is bound provided that

$$\cos \beta > \frac{a}{b}.$$

Show, further, that the amplitude A of the motion is given approximately by

$$A = \frac{1 - \cos \beta}{b \cos \beta - a}. \quad [5]$$