Question 4

A bead of mass m slides smoothly under the influence of gravity on a wire with equation

$$z = -a\cos x, \qquad -\frac{\pi}{2} \le x \le \frac{\pi}{2},$$

where the z-axis is vertically upwards and a is a positive constant. The wire stops at $x = \pm\pi/2$, and if $|x| > \pi/2$ the particle moves freely under gravity. The wire rotates about the z-axis with constant angular velocity Ω.

(i) Show that the Hamiltonian of the particle for $|x| < \pi/2$ is

$$H(x,p) = \frac{p^2}{2m(1 + a^2\sin^2 x)} - \frac{1}{2}m\Omega^2 x^2 - mga\cos x. \qquad [5]$$

(ii) Show that the system has a fixed point at $x = p = 0$, and that this is stable if $ga > \Omega^2$, but unstable if $ag < \Omega^2$. [5]

(iii) Show that if $1 < ga/\Omega^2 < \pi/2$, there is a stable fixed point at some point $x = x_1$ where $1 < x_1 < \pi/2$, but if $ga/\Omega^2 > \pi/2$ there are no stable fixed points. [5]

(iv) Show that the equation of the separatrix through the origin is

$$p_s(x) = m\sqrt{(1 + a^2\sin^2 x)\,(\Omega^2 x^2 - 4ga\sin^2(x/2))},$$

and deduce that, in the case $ga/\Omega^2 > \pi/2$, a particle starting on the separatrix at $x = b < \pi/2$ takes a time

$$T(b) = \int_b^{\pi/2} dx \sqrt{\frac{1 + a^2\sin^2 x}{\Omega^2 x^2 - 4ga\sin^2(x/2)}}$$

to fly off the end of the wire. [10]