Question 2

(a) Determine the values of the constants α and β such that the transformation $(q,p) \to (Q,P)$ given by

$$Q = (\alpha^2 - p^2)\, e^{-q}, \qquad P = \frac{p\beta e^q}{\alpha^2 - p^2},$$

is canonical. By expressing q as

$$q = \ln(\alpha - p) + \ln(\alpha + p) - \ln Q,$$

find an $F_3(Q,p)$ generating function for this transformation. Using this generating function, find $F_1(Q,q)$. [10]

(b) A canonical transformation $(q,p) \to (Q,P)$ can be described by the two generating functions $F_2(P,q)$ and $F_1(Q,q)$ where

$$F_2(P,q) = F_1(Q,q) + PQ, \qquad \text{where} \quad P = -\frac{\partial F_1}{\partial Q}.$$

By taking (P,q) as the independent variables and by differentiating this expression with respect to q and using the chain rule, show that

$$\frac{\partial F_2}{\partial q} = \frac{\partial F_1}{\partial q}.$$ [3]

By further differentiation show that

(i) $\displaystyle \frac{\partial^2 F_2}{\partial P \partial q} = \frac{\partial^2 F_2}{\partial P^2}\frac{\partial^2 F_1}{\partial q \partial Q},$

(ii) $\displaystyle \frac{\partial^2 F_1}{\partial Q \partial q} = -\frac{\partial^2 F_1}{\partial Q^2}\frac{\partial^2 F_2}{\partial P \partial q},$

(iii) $\displaystyle \frac{\partial^2 F_2}{\partial q^2} = \frac{\partial^2 F_1}{\partial q^2} + \frac{\partial^2 F_2}{\partial P \partial q}\frac{\partial^2 F_1}{\partial q \partial Q}.$ [12]

Question 3

(a) Sketch the phase curves for a particle of mass m moving in the potential

$$V(q) = \begin{cases} \infty, & q < 0, \\ a q^{1/2}, & q > 0, \end{cases}$$

where a is a positive constant. [5]

(b) Show that the action variable, I, and energy, E, are related by

$$E = \left(\frac{15\pi a^2}{8\sqrt{2m}}\right)^{2/5} I^{2/5},$$

and find an expression for the frequency of the motion as a function of the energy. [8]

(c) Show that the angle variable θ, such that $q = 0$ when $\theta = 0$, is given by

$$\theta(q) = \pi\left(1 - \tfrac{9}{8}\cos\phi + \tfrac{1}{8}\cos 3\phi\right), \qquad \phi = \sin^{-1}\left(\frac{a^2 q}{E^2}\right)^{1/4},$$

where $0 \le q \le (E/a)^2$ and $p \ge 0$. [8]

(d) How does θ depend upon q when $0 \le q \le (E/a)^2$ and $p \le 0$? [4]