

### Question 2

- (a) Determine the values of the constants  $\alpha$  and  $\beta$  such that the transformation  $(q, p) \rightarrow (Q, P)$  given by

$$Q = (\alpha^2 - p^2) e^{-q}, \quad P = \frac{p\beta e^q}{\alpha^2 - p^2},$$

is canonical. By expressing  $q$  as

$$q = \ln(\alpha - p) + \ln(\alpha + p) - \ln Q,$$

find an  $F_3(Q, p)$  generating function for this transformation. Using this generating function, find  $F_1(Q, q)$ . [10]

- (b) A canonical transformation  $(q, p) \rightarrow (Q, P)$  can be described by the two generating functions  $F_2(P, q)$  and  $F_1(Q, q)$  where

$$F_2(P, q) = F_1(Q, q) + PQ, \quad \text{where } P = -\frac{\partial F_1}{\partial Q}.$$

By taking  $(P, q)$  as the independent variables and by differentiating this expression with respect to  $q$  and using the chain rule, show that

$$\frac{\partial F_2}{\partial q} = \frac{\partial F_1}{\partial q}. \quad [3]$$

By further differentiation show that

$$\begin{aligned} \text{(i)} \quad \frac{\partial^2 F_2}{\partial P \partial q} &= \frac{\partial^2 F_2}{\partial P^2} \frac{\partial^2 F_1}{\partial q \partial Q}, \\ \text{(ii)} \quad \frac{\partial^2 F_1}{\partial Q \partial q} &= -\frac{\partial^2 F_1}{\partial Q^2} \frac{\partial^2 F_2}{\partial P \partial q}, \\ \text{(iii)} \quad \frac{\partial^2 F_2}{\partial q^2} &= \frac{\partial^2 F_1}{\partial q^2} + \frac{\partial^2 F_2}{\partial P \partial q} \frac{\partial^2 F_1}{\partial q \partial Q}. \end{aligned} \quad [12]$$

### Question 3

- (a) Sketch the phase curves for a particle of mass  $m$  moving in the potential

$$V(q) = \begin{cases} \infty, & q < 0, \\ aq^{1/2}, & q > 0, \end{cases}$$

where  $a$  is a positive constant. [5]

- (b) Show that the action variable,  $I$ , and energy,  $E$ , are related by

$$E = \left( \frac{15\pi a^2}{8\sqrt{2m}} \right)^{2/5} I^{2/5},$$

and find an expression for the frequency of the motion as a function of the energy. [8]

- (c) Show that the angle variable  $\theta$ , such that  $q = 0$  when  $\theta = 0$ , is given by

$$\theta(q) = \pi \left( 1 - \frac{2}{3} \cos \phi + \frac{1}{3} \cos 3\phi \right), \quad \phi = \sin^{-1} \left( \frac{a^2 q}{E^2} \right)^{1/4},$$

where  $0 \leq q \leq (E/a)^2$  and  $p \geq 0$ . [8]

- (d) How does  $\theta$  depend upon  $q$  when  $0 \leq q \leq (E/a)^2$  and  $p \leq 0$ ? [4]