Question 4

(a) A charged particle A is fixed at the lowest point of a circular wire of radius R held rigidly in a vertical plane. Another similar charged particle B of mass m is free to move on the circle. Assuming that the potential energy between the two charges is $V = e^2/L$, where L is the distance AB and e is the charge of each particle, show that the Hamiltonian of the particle B is

$$H(\theta, p) = \frac{p^2}{2mR^2} + \frac{e^2}{2R\sin(\theta/2)} + mgR(1-\cos\theta), \qquad 0 < \theta < 2\pi,$$

where θ is the angle between the radii of the two particles. [6]

(b) Show that if $m < e^2/(8gR^2)$, then there is only one fixed point, at $\theta = \pi$, $p = 0$, and that it is stable. [4]

Show, further, that if $m > e^2/(8gR^2)$, then there are three fixed points, one at $\theta = \pi$, $p = 0$ which is unstable and the others at $p = 0$ and the roots of

$$\sin^3(\theta/2) = \frac{e^2}{8mgR^2},$$

and that these are stable. [4]

Sketch the phase curves in the case of three fixed points. [4]

(c) In the small mass case, $m \le e^2/(8gR^2)$, show that the period of the motion with energy E can be expressed in the form

$$T(E) = 4R\sqrt{2m}\int_0^{x_1} dx\, \frac{1}{\sqrt{E - \frac{e^2}{2R\cos(x/2)} - 2mgR\cos^2(x/2)}},$$

where $x_1(E)$ is the positive turning point of the librational motion. [2]

(d) Show that in the special case $m = e^2/(8gR^2)$, the period of small-amplitude oscillations can be written in the form

$$T(E) = \frac{4R\sqrt{2m}}{(\varepsilon\beta)^{1/4}}\int_0^1 \frac{dy}{\sqrt{1-y^4}},$$

where

$$\varepsilon = E - \frac{e^2}{2R} - 2mR, \qquad \beta = \frac{mgR}{24}\left(1 + \frac{5e^2}{32mgR^2}\right).$$ [5]