

(Covering Units 5, 6 and 7 of Block II.)

Each question is marked out of 25. Your overall grade will be determined by the sum of your marks for all four questions.

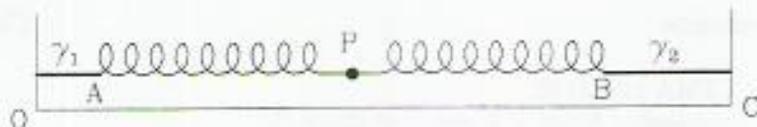
Question 1

- (a) Find the Hamiltonian corresponding to the Lagrangian

$$L(q, \dot{q}) = q^2 \left(\frac{1}{2} \dot{q}^2 - 1 \right), \quad q > 0.$$

Write down Lagrange's equation of motion for this Lagrangian, and confirm that it is equivalent to Hamilton's equations for the Hamiltonian you obtain. [8]

- (b)



A particle P of mass m moves in a straight line on a smooth horizontal surface under the influence of two perfect springs AP and PB of natural length l_0 and stiffness k^2 . The ends A and B of the springs are made to oscillate about the stationary points O and C respectively, such that $OA = \gamma_1(t)$ and $BC = \gamma_2(t)$, where γ_1 and γ_2 are known functions of time.

If the distance OC is d and the distance of the mass P from the oscillating point A is x , show that the Lagrangian function $L(x, \dot{x}, t)$ can be written as

$$L(x, \dot{x}, t) = \frac{1}{2} m \dot{x}^2 - k^2 x^2 - x \{ k^2 (\gamma_1 + \gamma_2 - d) + m \dot{\gamma}_1 \}. \quad [14]$$

[Hint: The force exerted by a spring of stiffness k^2 when its extension is e has magnitude $k^2 e$.]

- (c) Using the Lagrangian obtained in part (b), find the equation of motion for the mass in terms of x . [3]

Question 2

- (a) Determine the values of the constants α and β such that the transformation $(q, p) \rightarrow (Q, P)$ given by

$$Q = \left(\frac{\alpha - p}{p} \right) e^{-q}, \quad P = p \beta e^q,$$

is canonical and, for these values, find an $F_1(Q, q)$ and an $F_2(P, q)$ generating function for it. [10]

- (b) Write down the Hamiltonian corresponding to the time-dependent Lagrangian of Question 1(b) in the case that $\gamma_1 = \gamma_2 = a \cos \Omega t$,

$$L(q, \dot{q}, t) = \frac{1}{2} m \dot{q}^2 - k^2 q^2 + dk^2 q + \alpha (m \Omega^2 - 2k^2) q \cos \Omega t.$$

Show that the transformation $(q, p) \rightarrow (Q, P)$ given by

$$Q = q - b(t), \quad P = p + a(t),$$

where $a(t)$ and $b(t)$ are functions of time only, is canonical, and find a suitable $F_2(P, q, t)$ generating function.