TMA MS323 01 Question 3: Specimen Solution

(a) Put $y = \dot{x}$ to obtain the system

$$\dot{x} = y, \qquad \dot{y} = -x - e^{2x}.$$

(Note that there are infinitely many ways of converting a pair of coupled first-order equations to a second-order equation; this is the 'obvious' way.)

The function $g(x) = x + e^{2x}$ approached $\pm\infty$ as $x \to \pm\infty$ and $g'(x) = 1 + 2e^{2x} > 0$ for all x. Thus the equation $g(x) = 0$ has only one real root, at $x = a$ say, and near here $g(x) = g(a) + (x-a)g'(a) + O(x-a)^2)$ with $g'(a) > 0$. Putting $u = x - a$ the linearised equations of motion are

$$\dot{u} = y, \qquad \dot{y} = -g'(a)u$$

so by comparison with Equation 3.37, Unit 3, page 68 (with $\nu = 1$, $\mu = 0$) we see that the linear system is a centre.

It is important to remember that the fixed point may not be a centre of the non-linear system – see the Linearisation Theorem of Subsection 3.6.2, page 79.

Differentiating the given expression gives

$$\frac{d}{dt}(\dot{x}^2 + x^2 + e^{2x}) = 2\dot{x}\left(\ddot{x} + x + e^{2x}\right) = 0,$$

so $\dot{x}^2 + x^2 + e^{2x}$ is constant along phase curves of the non-linear system.

Let $f(x,y) = y^2 + x^2 + e^{2x}$. Then $f(x,y)$ is constant along phase curves in phase space.

On a phase curve near the fixed point $f(x,y)$ can be expanded using Taylor's expansion.

$$f(x,y) = f(a,0) + \tfrac{1}{2}y^2 + (x-a)^2 g'(a) + \cdots$$

so the lines defined by the equation $f(x,y) = c$ =constant, where $c \simeq f(a,0)$ are closed, because $g'(a) > 0$, and the fixed point must be a centre.

(b) At a fixed point of the system we have either $x = 2$ and

$$y^2 + 2y - 3 = (y+3)(y-1) = 0;$$

or $x = -2$ and

$$y^2 - 2y - 3 = (y-3)(y+1) = 0.$$

The fixed points are thus

$$(-2,3), \quad (-2,-1), \quad (2,-3), \quad (2,1).$$

The Jacobian matrix is

$$\mathsf{A}(x,y) = \begin{bmatrix} 2x & 0 \\ y & 2y+x \end{bmatrix}. \quad ?$$

The fixed points are classified (and sketched) below.

At $\mathbf{x} = (-2,3)$,

$$\mathsf{A}(-2,3) = \begin{bmatrix} -4 & 0 \\ 3 & 4 \end{bmatrix}, \qquad \lambda = -4, 4.$$

The fixed point is a saddle.

At $\mathbf{x} = (-2,-1)$,

$$\mathsf{A}(-2,-1) = \begin{bmatrix} -4 & 0 \\ -1 & -4 \end{bmatrix}, \qquad \lambda = -4, -4$$

The fixed point is a stable star.

The motion near the remaining two fixed points can be deduced from the above results since the transformation $u = -x$, $v = -y$ and $\tau = -t$ leaves the equations of motion unchanged in form. Therefore the phase curves near $(2,-3)$ have the same shape as those near $(-2,3)$, but the direction of the arrows is reversed; thus this fixed point is also a saddle.

Similarly for the fixed points at $(2,1)$ and $(-2,-1)$; thus $(2,1)$ is also an unstable star.

These results may also be obtained by direct calculation, of course.