$$T(E) = \sqrt{2}\int_{-\alpha}^{\alpha} dx \frac{\sqrt{1+a\sin x}}{\sqrt{E+\cos x}}$$

$\sin x = 2\sin^2 x/2 \cos x/2$,

$\cos x = 1 - 2\sin^2 x/2$

$$\therefore T(E) = \sqrt{2}\int_{-\alpha}^{\alpha} dx \frac{\sqrt{1+\frac{1}{2}a\sin^2 x/2 \cos x/2}}{\sqrt{E+1-2\sin^2 x/2}}$$

For small oscillations of the order $\alpha = \delta$, we can put $\sin x/2 = \delta \sin u$

$-\pi/2 \le u \le \pi/2$ ($x = \delta \sin \delta/2 \simeq \delta/2 \sin u \Rightarrow 1 \simeq \sin u \Rightarrow u = \pi/2$;
$x = -\delta \sin \delta/2 \simeq \delta/2 \sin u \Rightarrow -1 \simeq \sin u \Rightarrow u = -\pi/2$)

Then $\frac{1}{2}\cos x/2 \frac{dx}{du} = \delta \cos u$

$$\therefore dx = \frac{2\delta \cos u\, du}{\cos x/2}$$

$$\cos x/2 = \sqrt{1-\sin^2 x/2} = \sqrt{1-\delta^2/4 \sin^2 u}$$

$$\therefore -dx = \frac{2\delta \cos u}{\sqrt{1-\delta^2/4\sin^2 u}}\, du$$

$$T(E) = \sqrt{2}\int_{-\pi/2}^{\pi/2} du \sqrt{\frac{1+\frac{1}{2}a\delta^2/2 \sin u \sqrt{1-\delta^2/4\sin^2 u}}{-1+\delta^2/2+1-2\delta^2/4\sin^2 u}} \frac{2\delta\cos u}{\sqrt{1-\delta^2/4\sin^2 u}}$$

$$= \sqrt{2}\int_{-\pi/2}^{\pi/2} du \sqrt{\frac{1+a\delta \sin u \sqrt{1-\delta^2/4\sin^2 u}}{\delta^2/2 - \delta^2/2 \sin^2 u}} \frac{\delta\cos u}{\sqrt{1-\delta^2/4\sin^2 u}}$$

$$= \sqrt{2}\int_{-\pi/2}^{\pi/2} du \sqrt{\frac{1+a\delta \sin u \sqrt{1-\delta^2/4\sin^2 u}}{\delta^2/2(1-\sin^2 u)}} \frac{\delta\cos u}{\sqrt{1-\delta^2/4\sin^2 u}}$$

$$= \sqrt{2}\int_{-\pi/2}^{\pi/2} du \frac{\sqrt{1+a\delta \sin u \sqrt{1-\delta^2/4\sin^2 u}}}{\delta/\sqrt{2}\cos u} \frac{\delta\cos u}{\sqrt{1-\delta^2/4\sin^2 u}}$$

$$= 2\int_{-\pi/2}^{\pi/2} du \sqrt{\frac{1+a\delta \sin u \sqrt{1-\delta^2/4\sin^2 u}}{1-\delta^2/4\sin^2 u}}$$

$$= 2\int_{-\pi/2}^{\pi/2} du \left(\left(1+\frac{\delta^2}{4}\sin^2 u + O(\delta^4)\right)\left(1+a\delta\sin u\left(1-\frac{\delta^2}{8}\sin^2 u + O(\delta^4)\right)\right)^{1/2}\right.$$

(Using the binomial expansion for $(1-\delta^2/4\sin^2 u)^{-1}$ and $\sqrt{1-\delta^2/4\sin^2 u}$)

$$= 2\int_{-\pi/2}^{\pi/2} du \left(\left(1+\frac{\delta^2}{4}\sin^2 u + O(\delta^4)\right)\left(1+a\delta\sin u - \frac{a\delta^3}{8}\sin^3 u + O(\delta^4)\right)\right)^{1/2}$$

$$= 2\int_{-\pi/2}^{\pi/2} \left(1 + a\delta\sin u - \frac{a\delta^3}{8}\sin^3 u + \frac{\delta^2}{4}\sin^2 u + \frac{a\delta^3}{4}\sin^3 u + O(\delta^4)\right)^{1/2}$$