Thus $(-2,-1)$ is strongly stable, $(2,-1)$ is strongly unstable, and the remaining two fixed points are unstable but not strongly so.

On the lines $x = \pm 2$ we have $\dot{x} = 0$ (and $\dot{y} = y^2 \pm 2y + 3$), so they are phase curves of the system, which are parallel to the y-axis.

TMA MS323 01 Question 4: Specimen Solution

(i) The fixed points are at the stationary points of the Hamiltonian, given by the roots of the equations

$$\frac{\partial H}{\partial q} = -\frac{ap^2 \cos q}{2(1 + a \sin q)^2} + \sin q = 0$$
$$\frac{\partial H}{\partial p} = \frac{p}{1 + a \sin q} = 0.$$

The second equation shows that $p = 0$ and then the first equation shows that $\sin q = 0$: that is the fixed points are at $(n\pi, 0)$, $n = 0, \pm 1, \pm 2, \cdots$.

At the extrema of the Hamiltonian there are stable fixed points and at the saddles there are unstable fixed points. Thus we need the second derivatives of the Hamiltonian in order to classify the fixed points.

$$\frac{\partial^2 H}{\partial q^2} = -\frac{ap^2}{2}\frac{d}{dq}\left(\frac{\cos q}{(1 + a \sin q)^2}\right) + \cos q = (-1)^n$$
$$\frac{\partial^2 H}{\partial q \partial p} = -\frac{ap \cos q}{(1 + a \sin q)^2} = 0$$
$$\frac{\partial^2 H}{\partial p^2} = \frac{1}{1 + a \sin q} > 0.$$

Thus for even values of n the Hamiltonian has local minima: these fixed points are centres. For odd values of n the Hamiltonian has saddles: these fixed points are saddles or hyperbolic fixed points.

(ii) The Hamiltonian is 2π-periodic in q, so we need only consider q in he range $-\pi \le q \le \pi$. The energy at the saddle $(\pm\pi, 0)$ is $E_s = -\cos \pi = 1$. Thus a contour of energy $E = E_s = 1$ passes through all the saddles. The equation of this contour is given by

$$\frac{p^2}{2(1 + a \sin q)} - \cos q = 1 \quad \text{or} \quad p(q) = \pm 2\sqrt{1 + a \sin q}\cos(q/2).$$

Near $q = p = 0$ the Taylor expansion of $H(q, p)$ is

$$H = \tfrac{1}{2}(p^2 + q^2) - 1,$$

so the minimum energy is $E = -1$ and all librational motion lies in the energy range $-1 < E < 1$.

For energies $-1 < E < 1$ the phase curves cross the q-axis perpendicularly as the equation $H(0, q) = E$, that is $\cos q = E$, has real roots at which $|dp/dq|$ is infinite. For these energies all motion is librational and the phase curves comprise closed loops around each centre.

For energies $-1 > E$ there is no motion. For energies $E > 1$ all motion is rotational as there are no real turning points. In the figure we show some representative phase curves in the upper half of the phase plane: as the Hamiltonian is an even function of p the phase curves in the lower half plane are obtained by reflection in the q-axis.