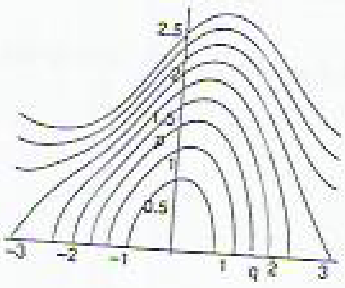

Figure 5 Figure showing some contours at energies $E = k/4$, $k = -2, \cdots, 6$, of the Hamiltonian $H = p^2/(2(1 + a\sin q) - \cos q$ with $a = 1/2$.

(iii) Because the phase curves are symmetric about the q-axis the period of any librational motion is

$$T(E) = 2\int_{q_1(E)}^{q_2(E)} \frac{dq}{\dot q} \quad \text{where} \quad \cos q_k = -E,\ k = 1, 2,$$

where $p > 0$ along the integration path. Also, from one of Hamilton's equations

$$\dot q = \frac{\partial H}{\partial p} = \frac{p}{1 + a\sin q},$$

and since the system is conservative

$$p = \sqrt{2(E + \cos q)(1 + a\sin q)} \quad \text{giving} \quad \dot q = \sqrt{\frac{2(E + \cos q)}{1 + q^2}}.$$

The integral for the period is thus

$$T(E) = \sqrt{2}\int_{q_1}^{q_2} dq\sqrt{\frac{1 + a\sin q}{E + \cos q}}, \quad \cos q_k = -E.$$

Since $q_1 < 0$ and $q_2 > 0$ define $\cos\alpha = -E$ and $0 \le \alpha < \pi$, so $q_1 = -\alpha$ and $q_2 = \alpha$, and

$$T(E) = \sqrt{2}\int_{-\alpha}^{\alpha} dq\sqrt{\frac{1 + a\sin q}{E + \cos q}}, \quad \cos\alpha = -E.$$

(iv) The period of the small amplitude librational motion with energy $E = -1 + \delta^2/2$, $0 < \delta \ll 1$ is obtained by expanding the integrand in powers of q. N=But first it is easiest to put $\cos q = 1 - 2\sin^2(q/2)$, giving $\sin(\alpha/2) = \delta/2$ and

$$T = 2\int_{-\alpha}^{\alpha} dq\sqrt{\frac{1 + a\sin x}{\frac{\delta^2}{4} - \sin^2(x/2)}} \quad \leftarrow \text{Correct}$$

Now define a new variable u, $\sin(x/2) = (\delta/2)\sin u$ to give

$$T = 2\int_{-\pi/2}^{\pi/2} du\sqrt{\frac{1 + a\delta\sin u\sqrt{1 - (\delta/2)^2\sin^2 u}}{1 - (\delta/2)^2\sin^2 u}}.$$

This is exact and now in a convienient form for expandingin powers of δ.

$$\begin{aligned} T &= 2\int_{-\pi/2}^{\pi/2} du\left(1 + \tfrac{1}{2}a\delta\sin u + \tfrac{1}{8}\delta^2\sin^2 u(1 - a^2) + \cdots\right) \\ &= 2\pi\left(1 + \tfrac{1}{16}\delta^2(1 - a^2) + \cdots\right). \end{aligned}$$