

Questions 2, 3 and 4 below, on *Units 2, 3 and 4* of Block I, form the second part of Tutor-marked Assignment MS323 01.

Each question is marked out of 25. Your overall grade will be based on the sum of your marks on these three questions and the question in Part 1.

Please send your answers to Questions 2 to 4 to your tutor. Your tutor should have kept the PT3 for this assignment so that there is no need to send another. (If your tutor has returned your original PT3 by mistake with your answer to Question 1, send it back with your answers to Questions 2 to 4.) You will eventually receive your copy of this PT3, completed by your tutor, along with your answers to these questions.

Question 2

- (a) Consider the first-order autonomous system with the velocity function

$$v(x) = x^3 - 3x + 2,$$

where x is allowed to take any real value.

- (i) Show that $x = -2$ is a fixed point, and locate all other fixed points. What are the elementary invariant sets of the system? Provide a sketch of its phase diagram. For each fixed point, state whether or not it is simple and whether it is stable, unstable or neither. [7]

- (ii) Provide a qualitative description of the motion for $t > 0$ of a phase point initially at $x = 3$. By considering the motion for large x , decide whether or not the motion terminates. [4]

- (iii) The system is perturbed by the addition to its velocity function of a small constant term (which is non-zero but may be either positive or negative). Discuss how the structure of the flow is changed, if at all, by the perturbation, mentioning in particular any change to the nature of the fixed points. [5]

- (b) Consider next the system with velocity function $\sqrt{v(x)}$, where x is now restricted to the range $-2 \leq x \leq 1$, and where $v(x)$ is the function of part (a) (and for any positive k , \sqrt{k} is the positive number whose square is k). [9]

Investigate the motions of this system, being careful to state where any natural boundaries are located, and paying particular attention to motion near natural boundaries and fixed points.

Question 3

- (a) Consider the second-order differential equation

$$\ddot{x} + x + \cos x = 0.$$

Convert it to a system of simultaneous first-order equations with $\dot{x} = y$. Show that the corresponding dynamical system has a single fixed point at $(a, 0)$, where $-\pi/2 < a < 0$, and whose linearisation is a centre.

Show that $\dot{x}^2 + x^2 + 2 \sin x$ remains constant along any phase curve of the system, and deduce the nature of the fixed point for the non-linear system. [7]

- (b) Consider the autonomous second-order non-linear system

$$\dot{x} = ax^2 - y^2, \quad \dot{y} = x^2 - by^2,$$

where a and b are positive real numbers. Show that the system has four fixed points, two of which are always saddles and that the other two are either a stable and an unstable spiral if $a < 4^{2/3}$ or a stable and an unstable node if $a > 4^{2/3}$. [18]