Question 4

Consider the system with the Hamiltonian

$$H(q,p) = \frac{p^2}{2(1+q^2)} - \cos q.$$

(i) Show that this system has saddle points at $(q,p) = ((2n+1)\pi, 0)$ and centres at $(q,p) = (2n\pi, 0)$, where $n = 0, \pm 1, \pm 2, \ldots$. [4]

(ii) Show that there is a contour with energy $E = 1$ joining all saddle points, and sketch the phase curves of the system, being careful to show the regions where the motion is librational. Show that all librational motion has energy in the range $-1 < E < 1$. [7]

(iii) Show that the period of the librational motion about the centre at $(2n\pi, 0)$, with energy E, can be expressed in terms of the integral

$$T(E) = \sqrt{2}\int_{-\alpha}^{\alpha} dx \sqrt{\frac{1+(2n\pi + x)^2}{E+\cos x}}, \qquad \text{with } \cos\alpha = -E,$$

where $0 \le \alpha < \pi$. [10]

(iv) Show that the small-amplitude librational motion about this centre has the frequency

$$\omega = \frac{1}{\sqrt{1+(2n\pi)^2}}.$$ [4]