Differentiating again gives

$$\frac{d^2x}{d\varepsilon^2} = -2e^a e^{-2x}\frac{dx}{d\varepsilon} = -2e^{2a}e^{-4x}.$$

For the third derivative we differentiate again,

$$\frac{d^3x}{d\varepsilon^3} = 8e^{2a}e^{-4x}\frac{dx}{d\varepsilon} = 8e^{3a}e^{-6x}.$$

At $x = a$ and $\varepsilon = 0$

$$\frac{dx}{d\varepsilon} = e^{-a}, \quad \frac{d^2x}{d\varepsilon^2} = -2e^{-2a}, \quad \frac{d^3x}{d\varepsilon^3} = 8e^{-3a},$$

which gives the required result when substituted into the Taylor series for $x(\varepsilon)$.

(b-ii) Taylor expansions of functions of two variables are summarised in *Unit 1*, Section 1.2.4. For this question we need to use Equation 1.8, page 14, so we need the first and second derivatives of $f(x,y) = \sinh(x-a) - ye^{-x}$,

$$\frac{\partial f}{\partial x} = \cosh(x-a) + ye^{-x}, \quad \frac{\partial f}{\partial y} = -e^{-x},$$

and

$$\frac{\partial^2 f}{\partial x^2} = \sinh(x-a) - ye^{-x}, \quad \frac{\partial^2 f}{\partial x \partial y} = e^{-x}, \quad \frac{\partial^2 f}{\partial y^2} = 0.$$

At $(a,0)$ these derivatives have the values

$$\frac{\partial f}{\partial x} = 1, \quad \frac{\partial f}{\partial y} = -e^{-a}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2} = 0, \quad \frac{\partial^2 f}{\partial x \partial y} = e^{-a}, \quad \frac{\partial^2 f}{\partial y^2} = 0.$$

so the second-order Taylor expansion of $f(x,y)$ about $(a,0)$ is

$$f(x,y) = (x-a) - ye^{-a} + (x-a)ye^{-a}.$$

The approximate root of the equation $f(x,y) = 0$ is therefore given by the equation

$$(x-a) - ye^{-a} + (x-a)ye^{-a} = 0 \quad \text{or} \quad x = a + \frac{ye^{-a}}{1+ye^{-a}}.$$

Since y is small this expression can be expanded using the Taylor expansion

$$(1+ye^{-a})^{-1} = 1 - ye^{-a} + O(y^2),$$

the last expression being obtained using the series given on page 11 of the *Handbook*. Thus we obtain

$$x = a + ye^{-a} - y^2e^{-2a} + O(y^3),$$

in agreement with the first three terms of the expansion found in part (bi).