TMA MS323 01 Question 1: Specimen Solution

(a) The Jacobian matrix of the transformation $(u(x,y), v(x,y))$ is defined in *Unit 1*, Section 1.4.6,

$$J(x,y) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix},$$

and a transformation is area-preserving if $\det(J(x,y)) = 1$ for all x and y.

(a-i) If $u = \frac{1}{2}x^2$ and $y = y^2/x$, then

$$\frac{\partial u}{\partial x} = x, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = -\frac{y^2}{x^2}, \quad \frac{\partial v}{\partial y} = \frac{2y}{x}.$$

and

$$J = \begin{bmatrix} x & 0 \\ -y^2x^{-2} & 2yx^{-1} \end{bmatrix}$$

so $\det(J) = 2y$ and the transformation is not area-preserving.

(a-ii) If $u = \tan x$ and $v = (y-k)\cos^2 x$, then

$$\frac{\partial u}{\partial x} = \frac{1}{\cos^2 x}, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = -2(y-k)\sin x\cos x, \quad \frac{\partial v}{\partial y} = \cos^2 x.$$

and

$$J = \begin{bmatrix} \cos^{-2} x & 0 \\ -2(y-k)\sin x\cos x & \cos^2 x \end{bmatrix}$$

so $\det(J) = 1$ and the transformation is area-preserving.

(b-i) Graphs of $y = \sinh(x-a)$ and $y = \varepsilon e^{-x}$, in the case $a = 1$ and $\varepsilon = 0.3$ are sketched in the figure

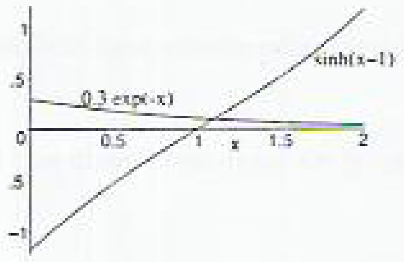

Figure 1 Graph of the functions $y = \sinh(x-1)$ and $y = 0.3e^{-x}$ showing that there is a root of the equation $\sinh(x-1) = 0.3e^{-x}$ near $x = a = 1$

These graphs suggest that for small ε, $\sinh(x-a) \simeq \varepsilon e^{-x}$ when $x = a$.

The given equation defines a function $x(\varepsilon)$ with $x(0) = \pi$ having a Taylor series expansion about the origin. Taylor expansions for functions of one variables are summarised in the *Handbook*, Section 1.6.

The derivatives of $x(\varepsilon)$ with respect to ε are most easily obtained by writing the equation in the form

$$\varepsilon = e^x \sinh(x-a) = \tfrac{1}{2}e^{-a}e^{2x} - \tfrac{1}{2}e^{a}.$$

differentiation with respect to ε then gives

$$1 = e^{2x}e^{-a}\frac{dx}{d\varepsilon}$$

which can be rearranged to give

$$\frac{dx}{d\varepsilon} = e^{a}e^{-2x}.$$