Question 3 – 25 marks

Consider the linear system

$$\ddot{x} + \omega(t)^2 x = 0$$

where the function $\omega(t)$ is a periodic function of time with period T, and

$$\omega(t)^2 = \begin{cases} a^2, & 0 \le t < \tau, \\ 0, & \tau \le t < T. \end{cases}$$

(a) Find the two solutions with the initial conditions

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and hence show that a fundamental matrix is

$$\Phi = \begin{pmatrix} \cos a\tau - a(T-\tau)\sin a\tau & \frac{1}{a}\sin a\tau + (T-\tau)\cos a\tau \\ -a\sin a\tau & \cos a\tau \end{pmatrix}. \qquad [10]$$

(b) Show that $\det(\Phi) = 1$ and that

$$\mathrm{Tr}(\Phi) = 2\cos\alpha - (\beta - \alpha)\sin\alpha, \quad \text{where} \quad \alpha = a\tau, \beta = aT. \qquad [5]$$

(c) In the case $\beta = 3\alpha$ show that, for $0 < \alpha < 2x_1$, where x_1 is the smallest positive root of $x \tan x = 1/2$, the system is stable. [10]

Question 4 – 8 marks

Show that the equation

$$\ddot{x} + (x^2 - a^2)\dot{x} + \tanh x = 0,$$

where a is a positive constant, has a periodic solution.

Now that you have completed your assignments for M821, please complete the course evaluation sheet which was sent to you with this *Assignment Booklet*. Please return it to the Courses Office, Faculty of Mathematics and Computing, as directed. Thank you.

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