Please make sure that the assignment number is correctly entered on your PT3 form as

$$\boxed{\text{M821 04}}.$$

Question 1 – 15 marks

(a) If $f(t)$ is any real, periodic function of principal period T and

$$\frac{dx}{dt} = xf(t)$$

show that, for all t, $x(t+T) = \mu x(t)$ where

$$\mu = e^{\rho T} \quad \text{where} \quad \rho = \frac{1}{T}\int_0^T dt\, f(t). \qquad [4]$$

(b) If $x(t)$ is a solution deduce, in each case, the conditions satisfied by $f(t)$ such that:

(i) $|x(t)|$ increases without bound as $t \to \infty$;

(ii) $x(t)$ is asymptotically stable;

(iii) $x(t)$ oscillates between finite, non-zero limits for all t. [6]

(c) Use the above results to determine the conditions on the positive constants a, b and T_1 such that the equation $\dot{x} = xf(t)$ has bounded solutions where

$$f(t) = \begin{cases} a, & 0 < t < T_1 < T, \\ -b, & T_1 \le t \le T, \end{cases}$$

and $f(t+T) = f(t)$ for all t. [5]

Question 2 – 20 marks

Assume that Liénard's equation

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0$$

has a T-periodic solution $p(t)$. Define $y(t)$ by the relation $x(t) = p(t) + y(t)$ and, by expanding to first-order in y, find the linear, second-order equation satisfied by $y(t)$. Show that this can be written as the pair of first-order equations

$$\frac{d\mathbf{z}}{dt} = \begin{pmatrix} 0 & 1 \\ -\dot{p}f'(p) - g'(p) & -f(p) \end{pmatrix}\mathbf{z}, \quad \mathbf{z} = \begin{pmatrix} y \\ \dot{y} \end{pmatrix}. \qquad [10]$$

Show that one solution of the equation satisfied by y is dp/dt and deduce that one of the characteristic numbers, μ_i, of the periodic orbits is unity and that the other is given by

$$\mu_2 = \exp\left(-\int_0^T dt\, f(p(t))\right). \qquad [10]$$