M821, TMA04 2001
Solution of question 2

Part a

Substituting a solution of the form $x(t) = p(t) + y(t)$ and expanding to first-order in y gives

$$\ddot{p} + \ddot{y} + (\dot{p} + \dot{y})\left(f(p) + yf'(p)\right) + g(p) + yg'(p) = 0.$$

Since $p(t)$ satisfies the original equation this reduces to

$$\ddot{y} + \dot{y}f(p) + \left(\dot{p}f'(p) + g'(p)\right)y = 0 \qquad (1) \quad \text{[5 marks]}$$

This is converted to a set of first-order equations by defining $z = \dot{y}$,

$$\frac{d}{dt}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\dot{p}f'(p) - g'(p) & -f(p) \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix}. \qquad (2) \quad \text{[5 marks]}$$

Part b

There are several ways to show that $\dot{p}$ is a solution of this equation. An elegant method is to note that since the original equation is autonomous $p(t+\epsilon)$ is also a solution for any constant ϵ. For small ϵ,

$$p(t+\epsilon) = p(t) + \epsilon\dot{p} + O(\epsilon^2).$$

It follows directly that $\dot{p}$ is a solution of equation 1.

To show that one of the characteristic numbers is unity observe that if $\Phi(t)$ is a fundamental matrix then $p(t)$ may be written in the form

$$p(t) = \Phi(t)\mathbf{c}$$

for some vector $\mathbf{c}$. Then, using the definition of E, JS page 246 equation (9.15),

$$p(t+T) = \Phi(t+T)\mathbf{c} = \Phi(t)E\mathbf{c}.$$

But, by definition, $p(t+T) = p(t) = \Phi(t)\mathbf{c}$ and hence

$$\Phi(t)E\mathbf{c} = \Phi(t)\mathbf{c} \quad \Longrightarrow \quad E\mathbf{c} = \mathbf{c}$$

since $\Phi(t)$ is nonsingular. Hence $\mathbf{c}$ is an eigenvector of E with eigenvalue $\mu = 1$, that is one of the characteristic numbers is unity.

The product of the characteristic numbers is given by Thereom 9.5 (JS page 250). Since there are two characteristic numbers and one is unity, this gives

$$\mu_2 = \exp\left(-\int_0^T dt\, f(p(t))\right), \quad \mu_1 = 1. \qquad \text{[10 marks]}$$