M821: TMA04 2001
Solution for question 1

Part (i)

The general solution of the differential equation is

$$x(t) = x(t_0)\exp\left(\int_{t_0}^{t} dt\, f(t)\right).$$

Thus $x(t+T) = \mu x(t)$ where $\mu = \exp\int_t^{t+T} dt\, f(t)$. Since $f(t)$ is periodic we have $\mu = \exp\int_0^T dt\, f(t)$ and so the characteristic exponent, defined by $e^{\rho T} = \mu$, JS page 248, equation (9.20), is

$$\rho = \frac{1}{T}\int_0^T dt\, f(t).$$

Part (ii)

The general solution of linear equations with periodic coefficients is given in equation (9.21) of JS. Here this takes the form $x = p(t)e^{\rho t}$, where $p(t)$ is a periodic function, and is therefore bounded. Thus

(a) if $|x(t)|$ increases without bound, $\rho > 0$;

(b) if $x(t)$ is asymptotically stable, that is $x(t) \to 0$ as $t \to \infty$, $\rho < 0$;

(c) if $x(t)$ oscillates between finite, non- zero limits, $\rho = 0$.

Part (iii)

In the particular case given

$$\rho = \frac{1}{T}\left\{a\int_0^{T_1} dt - b\int_{T_1}^{T} dt\right\} = \frac{1}{T}\left\{aT_1 - b(T - T_1)\right\}.$$

Thus, from part (ii), the solutions are bounded if $\rho \leq 0$, or

$$T_1 \leq \frac{bT}{a+b}.$$