M821: TMA04 2001
Solution for question 3

Part i) The equation of motion is

$$\ddot{x} + a^2 x = 0 \quad \text{for} \quad 0 \le t < \tau$$

with general solution $x(t) = A\cos at + B\sin at$, and

$$\ddot{x} = 0 \quad \text{for} \quad \tau < t < T$$

with general solution $x(t) = C + D(t-\tau)$, where the constants A, B, C and D are connected because the solution and its derivatives must be continuous at $t = \tau$.

For the particular initial conditions given this gives

$$x(t) = \begin{cases} \cos at, & 0 \le t < \tau, \\ \cos a\tau - a(t-\tau)\sin a\tau & \tau < t < T, \end{cases}$$

and

$$x(t) = \begin{cases} \frac{1}{a}\sin at & 0 \le t < \tau, \\ \frac{1}{a}\sin a\tau + (t-\tau)\cos a\tau & \tau < t < T, \end{cases}$$

On putting $t = T$ we obtain the value of a fundamental matrix at $t = T$,

$$\Phi = \begin{pmatrix} \cos a\tau - a(T-\tau)\sin a\tau & \frac{1}{a}\sin a\tau + (T-\tau)\cos a\tau \\ -a\sin a\tau & \cos a\tau \end{pmatrix}.$$

Part ii) We have

$$\det(\Phi) = (\cos a\tau - a(T-\tau)\sin a\tau)\cos a\tau + \left(\frac{1}{a}\sin a\tau + (T-\tau)\cos a\tau\right) a\sin a\tau = 1$$

and

$$\begin{aligned} \mathrm{Tr}(\Phi) &= 2\cos\alpha - (\beta - \alpha)\sin\alpha \\ &= 2\cos\alpha - 2\alpha\sin\alpha \quad \text{when} \quad \beta = 3\alpha \end{aligned}$$

where $\alpha = a\tau$ and $\beta = aT$.

Part iii) The eigenvalues, μ, of the fundamental matrix Φ are given by the solutions of the equation

$$\mu^2 - \mathrm{Tr}(\Phi)\mu + \det(\Phi) = 0$$

and because $\det(\Phi) = 1$ we have $\mu_1\mu_2 = 1$, so if the eigenvalues are real one is greater than unity and the system is unstable. If they are complex they have