Question 3 – 25 marks

In Problem 7 of Chapter 8 of the Course Notes, you showed that the Mandelbrot set, M, is contained in $\{c : |c| \leq 2\}$, implying $M \cap \mathbf{R} \subset [-2, 2]$. The aim of parts (a) and (b) of this question is to show that $M \cap \mathbf{R} = [-2, 1/4]$. (Note that, throughout this question, f_c denotes the function defined by $f_c(z) = z^2 + c$.)

(a) Let $c \in (1/4, \infty)$. Show that, for each $z \in \mathbf{R}$, $f_c^k(z) \to \infty$ as $k \to \infty$ and hence deduce that $c \notin M$.

(b) Let $c \in [-2, 1/4]$. Show that

$$|f_c(0)| \leq f_c\left(1/2 + \sqrt{1/4 - c}\right) = 1/2 + \sqrt{1/4 - c}.$$

Deduce that f_c maps the interval $[-1/2 - \sqrt{1/4 - c}, 1/2 + \sqrt{1/4 - c}]$ into itself and hence deduce that $c \in M$.

(c) Let $f : \mathbf{C} \to \mathbf{C}$ be defined by $f(z) = 3z^2 + 2iz + (1 - i)/3$. Use the fact that $M \cap \mathbf{R} = [-2, 1/4]$ to determine whether or not $J(f)$ is connected.

Question 4 – 25 marks

Let $f : \mathbf{C} \to \mathbf{C}$ be defined by $f(z) = z^4 + c$, where $|c| > 2$. Let C be the circle $\{z : |z| = |c|\}$ and D be its interior $\{z : |z| < |c|\}$. Then

$$f^{-1}(C) = \{(ce^{i\theta} - c)^{1/4} : 0 \leq \theta \leq 8\pi\}$$

is made up of four loops passing through 0 with no other points of intersection. The interior of each of the four loops is mapped by f bijectively onto D.

(a) Show that each of the four branches of f^{-1} maps $A = \{z : |z| \leq |2c|^{1/4}\}$ into itself and is a contraction on A.

(b) It follows from Theorem 9.1 of Falconer that there is a unique non-empty compact set $F \subset A$ that is invariant for the contractions identified in part (a). Show that F is equal to the Julia set, $J(f)$.

(c) Hence obtain an estimate for $\dim_H J(f)$ for large values of $|c|$.

Now that you have completed your assignments for M835, please complete the course evaluation sheet which was sent to you with this *Assignment Booklet*. Please return it to the Courses Office, Faculty of Mathematics and Computing, as directed. Thank you.

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