$\therefore -1 \leq \inf\{f(x): 0 < x < r\} \leq \cancel{f(x_n)} = -1$ ✓

$\therefore \underline{\lim}_{x \to 0} f(x) = \lim_{r \to 0}(\inf\{f(x): 0 < x < r\}) = -1$ ✓

For the upper limit we need to consider rational approximations to $\frac{1}{2\pi n} = x$ for each n. We can take the 1st m digits then if we call this approximation $x_m(n)$

$|x_m(n) - 1/2\pi n| \leq 0.5 \times 10^{-m} \leq 10^{-m} = \delta$. (1)

From continuity of cos; for $\varepsilon > 0$ we can find $\delta > 0$ such that $|x - y| < \delta \Rightarrow |\cos x - \cos y| < \varepsilon$.

hence so

$-\varepsilon < \cos(1/x_m(n)) - \cos 2\pi n < \varepsilon$

$1 - \varepsilon < \cos(1/x_m(n)) < 1 + \varepsilon$.

Since $\cos(y) \leq 1$ we can write

$1 - \varepsilon < \cos(1/x_m(n)) \leq 1$. (2)

Let ε_k be a sequence tending to zero then for each ε_k we can find a rational approximation $x_m(n)$ to $1/2\pi n$ such that (1) implies (2) then $1 - \varepsilon_k < \cos(1/x_m(n)) \leq 1$.

It follows that as $k, m, n \to \infty$

4/6 $\overline{\lim}_{x \to 0} f(x) = \lim_{\delta \to 0}(\sup\{f(x): 0 < x < \delta\}) = 1$. 8/15

r was used in the course notes

For each $r > 0$, $\exists\, n \in \mathbb{N}$ such that $0 < \frac{1}{2n\pi} < r$, and hence

$\sup\{\cos\left(\frac{1}{x}\right) : 0 < x < r\} = 1$ so,

$\overline{\lim}_{x \to 0} F(x) = \lim_{r \to 0}(\sup\{F(x): 0 < x < r\}) = 1$