P6578708 for some $\xi \in [-1, 1]$. (15)

$$Lf = \int_{-1}^{1} K(\theta) f^{3}(\theta)\, d\theta \leq \int_{-1}^{1} K(\theta)\, d\theta\, f'''(\xi), \text{ by MVT}$$

$$= \frac{f'''(\xi)}{2} \int_{-1}^{1} \left((-1-\theta)_+^2 + 2(-\theta)_+^2 - 3(1-\theta)_+^2 + 8(1/4-\theta)_+ \right) d\theta$$

On $[-1, 0]$ $K(\theta) = -3(1-\theta)^2 + 8(1/4 - \theta) + 2\theta^2$

$$= -3 + 6\theta - 3\theta^2 + 2 - 8\theta + 2\theta^2$$
$$= -\theta^2 - 2\theta - 1 = -(\theta+1)^2$$

On $[0, 1/4]$ $K(\theta) = -3(1-\theta)^2 + 8(1/4 - \theta)$
$$= -3 + 6\theta - 3\theta^2 + 2 - 8\theta$$
$$= -3\theta^2 - 2\theta - 1, \text{ and this}$$
does not change sign on $[0, 1/4]$

On $[1/4, 1]$ $K(\theta) = -3(1-\theta)^2$.

$$Lf \leq \frac{f^3(\xi)}{2}\left(\int_{-1}^{0} -(\theta+1)^2\, d\theta + \int_{0}^{1/4} -(3\theta^2 + 2\theta + 1)\, d\theta + \int_{1/4}^{1} -3(1-\theta)^2\, d\theta \right)$$

$$= -\frac{f^3(\xi)}{2}\left(\left[\frac{1}{3}(\theta+1)^3\right]_{-1}^{0} + \left[\theta^3 + \theta^2 + \theta\right]_{0}^{1/4} + \left[-(1-\theta)^3\right]_{1/4}^{1} \right)$$

$$= -\frac{f^3(\xi)}{2}\left(\frac{1}{3} - 0 + \frac{1}{64} + \frac{1}{16} + \frac{1}{4} - 0 + 0 + \frac{27}{64} \right)$$

$$= -\frac{f^3(\xi)}{384}\left(64 + 3 + 12 + 48 + 81 \right)$$

$$= -\frac{208}{384} f'''(\xi) = -\frac{13 f'''(\xi)}{24}$$

(20)