The formula given in Powell for the Chebyshev polynomials is missing a constant. The correct expression is

$$T_n(x) = (-1)^n\left[\prod_{k=0}^{n-1}(2(n-k)-1)\right]^{-1}(1-x^2)\frac{d^n}{dx^n}\left((1-x^2)^{n-\frac{1}{2}}\right)$$

$$= K(1-x^2)\frac{d^n}{dx^n}\left((1-x^2)^{n-\frac{1}{2}}\right)$$

where K is a constant.

$$\therefore U_n = \frac{(T_{n+1})'}{n+1}$$

$$= \frac{K}{n+1}\frac{d}{dx}\left((1-x^2)\frac{d^{n+1}}{dx^{n+1}}\left((1-x^2)^{n+\frac{1}{2}}\right)\right)$$

$$= \frac{(-1)^n}{n+1}\left[\prod_{k=0}^{n-1}(2(n-k)-1)\right]^{-1}\frac{d}{dx}\left((1-x^2)\frac{d^{n+1}}{dx^{n+1}}\left((1-x^2)^{n+\frac{1}{2}}\right)\right).$$

This is ~~easier~~ more easily proved directly.

$$\frac{d}{d\theta}\left(T_n(\cos\theta)\right) = T_n'(\cos\theta)(-\sin\theta)$$

$$= \frac{d}{d\theta}(\cos n\theta) = -n\sin n\theta$$

$$\therefore T_n'(\cos\theta) = n\frac{\sin n\theta}{\sin\theta} = nU_n(\cos\theta)$$

etc.