$6\frac{1}{2}$

b) $\frac{d T_{n+1}(\cos\theta)}{dx} = \frac{d\theta}{dx}\frac{d T_{n+1}(\cos\theta)}{d\theta} = \frac{d\theta}{dx}\frac{d(\cos(n+1)\theta)}{d\theta}$

$x = \cos\theta$

$\therefore dx = -\sin\theta \, d\theta$

So $\frac{d\theta}{dx} = \frac{-1}{\sin\theta}$

$\therefore$ And $\frac{d(\cos(n+1)\theta)}{d\theta} = -(n+1)\sin(n+1)\theta$

Then $T'_{n+1} = \frac{-1}{\sin\theta} \times -(n+1)\sin(n+1)\theta$

$= \frac{(n+1)\sin(n+1)\theta}{\sin\theta}$

$= (n+1)U_n$ as required.

c) f, g orthogonal $\Rightarrow \int_a^b f g \, \omega \, dx = 0$

$\int_{-1}^{1} U_n U_m \, \omega \, dx = 0 \qquad n \neq m.$

Put $x = \cos\theta$, $dx = -\sin\theta \, d\theta$

$x = 1 \Rightarrow \theta = 0 \quad x = -1 \Rightarrow \theta = \pi$

$\int_\pi^0 \frac{\sin(n+1)\theta}{\sin\theta}\frac{\sin(m+1)\theta}{\sin\theta}(-\sin\theta)\sqrt{1-\cos^2\theta}\, d\theta$

$= \int_0^\pi \sin(n+1)\theta \sin(m+1)\theta \, d\theta$

$= \frac{1}{2}\int_0^\pi \cos(n-m)\theta - \cos(n+m+2)\theta \, d\theta$

$= \frac{1}{2}\left[\frac{\sin(n-m)\theta}{n-m} - \frac{\sin(n+m+2)\theta}{n+m+2}\right]_0^\pi$

$= 0$. $\therefore U_n, U_m$ orthogonal on $[-1, 1]$ wrt $\sqrt{1-x^2}$.