$\cos(n+1)\theta \sin\theta = \cos\theta \sin(n+1)\theta - \sin n\theta$
Hence
$\sin n\theta = \cos\theta \sin(n+1)\theta - \cos(n+1)\theta\sin\theta$
$\sin((n+1)\theta - \theta) = \cos\theta\sin(n+1)\theta - \cos(n+1)\theta\sin\theta$
Which is a standard formula.
Hence U_n also satisfies the recurrence relation.

$\psi_0 = 0$ $\psi_1 = \pi$ $\psi_2 = 2\pi x$ $\psi_3 = \pi(4x^2-1)$

U $\quad U_0 = 1 \quad U_1 = 2x \quad U_2 = \dfrac{\sin 3\theta}{\sin\theta}$

$= \dfrac{3\sin\theta - 4\sin^3\theta}{\sin\theta}$

$= 3 - 4\sin^2\theta$
$= 3 - 4 + 4\cos^2\theta$
$= 4x^2 - 1$
where $x = \cos\theta$

$\therefore$ for $i = 0, 1, 2$, $\psi_{n+1} = \pi U_n$

Hypothesis: $\psi_{n+1} - \pi U_n = 0$ for all n

$\psi_{n+1} = 2x\psi_n - \psi_{n-1}$ ①

$\pi U_n = 2x\pi U_{n-1} - \pi U_{n-2}$ ②.

① − ②

$\psi_{n+1} - \pi U_n = 2x(\psi_n - \pi U_{n-1}) - (\psi_{n-1} - \pi U_{n-2})$

RHS ~~zero~~ zero for $n = 2$, $\therefore$ LHS zero $\Rightarrow \psi_3 = \pi U_2$

of course, we know this. Suppose
$\psi_{n+1} = \pi U_n$ for $n = 0, 1, \ldots . k$
Try $n = k+1$
$\psi_{k+2} - \pi U_{k+1} = 2x(\psi_{k+1} - \pi U_k) - (\psi_k - \pi U_{k-1})$
$= 0$

$\therefore$ Hypothesis true by induction