Please make sure that the assignment number is correctly entered on your PT3 form as

$\boxed{\text{M832 03}}$.

Give any numerical answers correct to 3 significant figures.

Question 1 (Chapter 11) – 20 marks

In this question we use the inner product

$$(f,g) = \int_0^1 x^2 f(x)\,g(x)\,dx, \qquad f,g \in \mathcal{C}[0,1].$$

(i) Determine the corresponding orthogonal functions ϕ_0, ϕ_1, ϕ_2, given by Theorem 11.3 (Powell, page 131).

(ii) Let $f(x) = x^{1/4}$, $0 \le x \le 1$. Use your answer to part (i) to find the function p^* in $\mathcal{P}_2$ which minimizes

$$\|f - p\|, \qquad p \in \mathcal{P}_2,$$

for the norm arising out of the above inner product, and determine the value of this minimum.

Question 2 (Chapter 12) – 40 marks

The aim of this question is to derive the Gaussian quadrature formula

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,dx \approx \frac{\pi}{k+1}\sum_{i=0}^{k} f(x_i), \qquad f \in \mathcal{C}[-1,1], \tag{$*$}$$

where x_i, $i = 0,1,\ldots,k$, are the zeros of the Chebyshev polynomial T_{k+1} (Powell, page 38).

(i) Let $w(x) = (1-x^2)^{-1/2}$, $-1 < x < 1$, and define

$$\psi_n(x) = \int_{-1}^{1} \frac{T_n(t) - T_n(x)}{t - x} w(t)\,dt, \quad -1 \le x \le 1, \quad n = 0,1,2,\ldots.$$

Prove that ψ_n satisfies the recurrence relation (4.25) (Powell, page 39), and deduce that

$$\psi_0(x) = 0, \quad \psi_1(x) = \pi, \quad \psi_2(x) = 2\pi x, \quad \psi_3(x) = \pi(4x^2 - 1).$$

(ii) The Chebyshev polynomials of the *second* kind are defined as

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \qquad -1 \le x \le 1, \qquad n = 0,1,2,\ldots,$$

where $x = \cos\theta$.

(a) Prove that U_n also satisfies (4.25), and deduce that

$$\psi_{n+1} = \pi U_n, \qquad n = 0,1,2,\ldots.$$

(b) Use the chain rule to prove that $T'_{n+1} = (n+1)U_n$, $n = 0,1,2,\ldots$.

(c) Show that the polynomials U_n, $n = 0,1,2,\ldots$, are orthogonal on $[-1,1]$ with respect to the weight function $\tilde{w}(x) = (1-x^2)^{1/2}$, and write down Rodrigue's formula for U_n.

(iii) Let $x_0, x_1, \ldots, x_k$ be the zeros of T_{k+1}, and define

$$\ell_i(x) = \prod_{\substack{j=0 \\ j \ne i}}^{k} \frac{x - x_j}{x_i - x_j}, \qquad -1 \le x \le 1, \qquad i = 0,1,\ldots,k.$$