$$= \int_{-1}^{1} \frac{\left(T_{n+1}(t) - 2x T_n(t) + T_{n-1}(t)\right) - \left(T_{n+1}(x) - 2x T_n(x) + T_{n-1}(x)\right)}{(t-x)\sqrt{1-t^2}}\,dt$$

The second bracketed term is zero; it is the recurrence relation for Chebyshev polynomials. The integral becomes

$$\int_{-1}^{1} \frac{T_{n+1}(t) - 2x T_n(t) + T_{n-1}(t)}{(t-x)\sqrt{1-t^2}}\,dt$$

Sub $t = \cos\theta$, $dt = -\sin\theta\, d\theta$
then $\sqrt{1-t^2} = \sin\theta$.

$$\int_{\pi}^{0} \frac{\cos(n+1)\theta - 2x\cos n\theta + \cos(n-1)\theta \cdot -\sin\theta\, d\theta}{(\cos\theta - x)\sin\theta}$$

(Note the limits have changed since have used defn. of Chebyshev polys. $T_n(\cos\theta) = \cos n\theta$.)

$$\int_0^{\pi} \frac{\cos(n+1)\theta - 2x\cos n\theta + \cos(n-1)\theta}{\cos\theta - x}\, d\theta.$$

$$\int_0^{\pi} \Big(\cos n\theta\cos\theta - \sin n\theta\sin\theta - 2x\cos n\theta$$

$$+ \cos n\theta\cos\theta + \sin n\theta\sin\theta\Big)\frac{d\theta}{\cos\theta - x}$$

$$= \int_0^{\pi} \frac{2\cos n\theta\cos\theta - 2x\cos n\theta}{\cos\theta - x}\, d\theta$$

$$= \int_0^{\pi} 2\cos n\theta\, d\theta = \left[\frac{2}{n}\sin n\theta\right]_0^{\pi} = 0$$

$\therefore\ \psi_n$ satisfies the recurrence relation.

$T_0(x) = 1$