$$\psi_0(x) = \int_{-1}^{1} \frac{T_0(t) - T_0(x)}{t-x}\, \omega(t)\, dt$$

$$= \int_{-1}^{1} \frac{(1-1)}{t-x}\, \omega(t)\, dt = 0 \text{ by defn.}$$

$\psi_1(x) = x$ etc

$$\psi_1(x) = \int_{-1}^{1} \frac{T_1(t) - T_1(x)}{t-x}\, \omega(t)\, dt$$

$$= \int_{-1}^{1} \frac{t-x}{t-x}\, \omega(t)\, dt = \int_{-1}^{1} \omega(t)\, dt$$

Sub $t = \cos\theta$, $dt = -\sin\theta\, d\theta$

$t = 1 \Rightarrow \theta = 0$, $t = -1 \Rightarrow \theta = \pi$

$$\psi_1(x) = \int_{-\pi}^{0} \frac{-\sin\theta\, d\theta}{\sqrt{1-\cos^2\theta}} = \int_{0}^{\pi} d\theta = \pi.$$

We can use the recurrence rel.

$$\psi_2(x) = 2x\psi_1(x) - \psi_0(x)$$
$$= 2x\pi - 0 = 2\pi x$$

$$\psi_3(x) = 2x\psi_2(x) - \psi_1(x)$$
$$= 2x(2\pi x) - \pi$$
$$= \pi(4x^2 - 1).$$

ii) a) We must prove

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$$

$$\frac{\sin(n+2)\theta}{\sin\theta} = \frac{2\cos\theta\sin(n+1)\theta}{\sin\theta} - \frac{\sin(n\theta)}{\sin\theta}$$

where $x = \cos\theta$

$$\frac{\sin(n+1)\theta\cos\theta + \cos(n+1)\theta\sin\theta}{\sin\theta} = \frac{2\cos\theta\sin(n+1)\theta}{\sin\theta} - \frac{\sin n\theta}{\sin\theta}$$

$$\frac{\cos(n+1)\theta\sin\theta}{\sin\theta} = \frac{\cos\theta\sin(n+1)\theta}{\sin\theta} - \frac{\sin n\theta}{\sin\theta}$$

$\theta \neq \pi n$. so we can multiply by $\sin\theta$ without ambiguity.