3

$$\therefore p_2^*(x) = \frac{12}{13} + \frac{80}{221}\left(x - \frac{3}{4}\right) \quad \frac{-60}{221}\left(x^2 - \frac{4x}{3} + \frac{2}{5}\right)$$

$$= \frac{120}{221} + \frac{160}{221}x - \frac{60}{221}x^2$$

$$\|f - p^*\|^2 = \int_0^1 x^2 (f - p^*)^2 \, dx$$

$$= \int_0^1 x^2 f^2 - 2fp^* x^2 + x^2 p^{*2} \, dx$$

$$= \int_0^1 x^{5/2} - 2x^{9/4}\left(\frac{120}{221} + \frac{160}{221}x - \frac{60}{221}x^2\right) + x^2\left(\frac{120}{221} + \frac{160}{221}x - \frac{60}{221}x^2\right)^2 dx$$

$$= \left[\frac{2}{7}x^{7/2}\right]_0^1 - \int_0^1 \frac{240}{221}x^{9/4} + \frac{320}{221}x^{13/4} - \frac{120}{221}x^{17/4} \, dx$$

$$\frac{1}{221^2}\int_0^1 120^2 x^2 + 160^2 x^4 + 60^2 x^6 + 240\times 160 x^3 - 120^2 x^4 - 160\times 120 x^5 \, dx$$

$$= 2/7 - \frac{1}{221}\left[\frac{960}{13}x^{13/4} + \frac{1280}{17}x^{17/4} - \frac{480}{21}x^{21/4}\right]_0^1$$

$$+ \frac{1}{221^2}\left[\frac{120^2}{3}x^3 + \frac{160^2}{5}x^5 + \frac{60^2}{7}x^7 + 60\times 160 x^4 - \frac{120^2}{5}x^5 - 160\times 20 x^6\right]_0^1$$

$$= 2/7 - \frac{195560}{7\times 221^2} + \frac{1}{221^2}\left(\frac{97680}{7}\right)$$

$$= \frac{1}{7}\times\frac{1}{221^2}\left(2\times 221^2 - 195560 + 97680\right)$$

Better expressed as a decimal

$$= 2/7 \times 221^2 \Rightarrow \|f - p^*\| = \frac{1}{221}\sqrt{2/7} = .0024186537 73..$$

2) i) We must prove that

$$\psi_{n+1} = 2x\psi_n - \psi_{n-1} \qquad (1)$$

holds

2a

$$\psi_n(x) = \int_{-1}^1 \frac{T_n(t) - T_n(x)}{t - x}\frac{1}{\sqrt{1-t^2}} \, dt.$$

From (1) $\psi_{n+1} - 2x\psi_n + \psi_{n-1} = 0$

$$0 = \int_{-1}^1 \frac{T_{n+1}(t) - T_{n+1}(x) - 2xT_n(t) + 2xT_n(x) + T_{n-1}(t) - T_{n-1}(x)}{(t-x)\sqrt{1-t^2}} \, dt$$