2

ii) $p^*(x) = \sum c_i \phi_i(x)$

The error function $f(x) - p^*(x)$ is orthogonal to ϕ_0, ϕ_1, ϕ_2.

ie $(f - p^*, \phi_i) = 0$

so $(f, \phi_i) - (p^*, \phi_i) = 0$, since $(\ \ )$ is linear.

$(f, \phi_i) - (\sum c_j \phi_j, \phi_i) = 0$

Since $(\ \ )$ is linear we can write

$(f, \phi_i) - \sum c_j (\phi_j, \phi_i) = 0$

$(\phi_j, \phi_i) = 0$ unless $j = i$ since the ϕ_i are all orthogonal.

$\therefore (f, \phi_i) - c_i(\phi_i, \phi_i) = 0$

$$\therefore c_i = \frac{(f, \phi_i)}{(\phi_i, \phi_i)}$$

$$c_0 = \frac{(f, \phi_0)}{(\phi_0, \phi_0)} = \frac{\int_0^1 x^2 \cdot x^{1/4}\,dx}{\int_0^1 x^2\,dx} = \frac{\left[\frac{4}{13} x^{13/4}\right]_0^1}{\left[\frac{1}{3}x^3\right]_0^1} = \frac{12}{13}$$

$$c_1 = \frac{(f, \phi_1)}{(\phi_1, \phi_1)} = \frac{\int_0^1 x^2 x^{1/4}(x - 3/4)\,dx}{\int_0^1 (x - 3/4)^2 x^2\,dx} = \frac{\int_0^1 x^{13/4} - \frac{3}{4}x^{9/4}\,dx}{\int_0^1 x^4 - \frac{3}{2}x^3 + \frac{9}{16}x^2\,dx}$$

$$= \frac{\left[4/17\, x^{17/4} - \frac{3}{13} x^{13/4}\right]_0^1}{\left[\frac{1}{5}x^5 - \frac{3}{8}x^4 + \frac{3}{16}x^3\right]_0^1} = \frac{4/17 - 3/13}{1/5 - 3/8 + 3/16} = \frac{80}{221}$$

$$c_2 = \frac{(f, \phi_2)}{(\phi_2, \phi_2)} = \frac{\int_0^1 x^2 x^{1/4}(x^2 - \frac{4}{3}x + 2/5)\,dx}{\int_0^1 x^2(x^2 - \frac{4x}{3} + 2/5)^2\,dx}$$

$$= \frac{\int_0^1 x^{17/4} - \frac{4}{3}x^{13/4} + \frac{2}{5}x^{9/4}\,dx}{\int_0^1 x^6 - \frac{8}{3}x^5 + \frac{4}{5}x^4 + \frac{16}{9}x^4 - \frac{16}{15}x^3 + \frac{4}{25}x^2\,dx}$$

$$= \frac{\left[\frac{4}{21}x^{21/4} - \frac{16}{51}x^{17/4} + \frac{8}{65}x^{13/4}\right]_0^1}{\left[\frac{1}{7}x^7 - \frac{4}{9}x^6 + \frac{4}{25}x^5 + \frac{16}{45}x^5 - \frac{4}{15}x^4 + \frac{4}{75}x^3\right]_0^1}$$

$$= \frac{-4/23205}{1/1575} = \frac{-4}{1} \times \frac{1575}{23205} = \frac{-4}{1} \times \frac{315}{4641}$$

$$= \frac{-4}{1} \times \frac{15}{221} = \frac{-60}{221}$$