Please make sure that the assignment number is correctly entered on your PT3 form as

$$\boxed{\text{M832 01}}.$$

Give any numerical answers correct to 3 significant figures.

Question 1 (Chapters 1 and 2) – 25 marks

(i) Determine the best approximations by a constant to the function

$$f(x) = \begin{cases} 0, & |x| \le 1-\varepsilon, \\ 1 + (|x|-1)/\varepsilon, & 1-\varepsilon < |x| \le 1, \end{cases}$$

where $0 < \varepsilon < 1$, with respect to the L_1, L_2 and L_∞ norms on $[-1, 1]$.

(ii) Determine the best approximation of the form $p(x) = \lambda x$ to a function $f(x) = a + bx$ in $\mathcal{P}_1$, with respect to the L_2 norm on $[0, 1]$. If this best approximation is denoted by $p = X(f)$, then prove that

$$\|X(f_1) - X(f_2)\|_2 \le \|f_1 - f_2\|_2, \qquad f_1, f_2 \in \mathcal{P}_1.$$

Question 2 (Chapter 3) – 25 marks

(i) Determine real numbers α, β, γ such that if $p \in \mathcal{P}_2$, then

$$p(2) = \alpha p(0) + \beta p(1) + \gamma p(3).$$

(ii) Suppose that $f \in C[0, 3]$ and that

$$f(2) = \alpha f(0) + \beta f(1) + \gamma f(3) + 0.08,$$

where α, β, γ are the numbers found in part (i). Prove that

$$\|f - p^*\|_\infty \ge 0.03,$$

where p^* is the best minimax approximation from $\mathcal{P}_2$ to f.

Question 3 (Chapter 4) – 25 marks

(i) Determine the Lagrange functions $l_k(x)$, $k = 0, 1, 2, 3, 4$, for the interpolation points $\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$ in $[-1, 1]$, and check that your answers satisfy equation (4.10) (Powell, page 35).

(ii) With $l_k(x)$ as in part (i), express the function

$$\sum_{k=0}^{4} |l_k(x)|$$

as a polynomial on each of the intervals $[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$. Check that your polynomials take the correct values at the interval end points.

(iii) Use your answers to part (ii) to confirm the second entry in the middle column of Table 4.5 (Powell, page 42).

Question 4 (Chapter 5) – 25 marks

(i) Use Newton's interpolation method to determine the polynomial in $\mathcal{P}_2$ which interpolates the function $f(x) = \log_e(1 + \frac{1}{2}x)$, $-1 \le x \le 1$, at the Chebyshev interpolation points given by equation (4.27) (Powell, page 39).

(ii) Determine an expression for the polynomial in $\mathcal{P}_5$ which interpolates the function values $f(0)$, $f(1)$, $f(2)$ and the derivative values $f'(0)$, $f'(2)$, $f''(2)$. Check that your calculation is correct by considering $f(x) = (x+1)^5$.