(8)

(18/25)

4) $\mathrm{Re}\left(\frac{t+z}{t-z}\right) = \mathrm{Re}\left(\frac{e^{i\phi}+re^{i\theta}}{e^{i\phi}-re^{i\theta}}\right) = \mathrm{Re}\left(\frac{e^{i\phi}+re^{i\theta}}{e^{i\phi}-re^{i\theta}} \times \frac{e^{-i\phi}-re^{-i\theta}}{e^{-i\phi}-re^{-i\theta}}\right)$

$= \mathrm{Re}\left(\frac{1-re^{i(\phi-\theta)}+re^{i(\theta-\phi)}-r^2}{1-re^{i(\phi-\theta)}-re^{i(\theta-\phi)}+r^2}\right)$

$= \mathrm{Re}\left(\frac{1-r^2+r\left(e^{i(\theta-\phi)}-e^{-i(\theta-\phi)}\right)}{1+r^2-r\left(e^{i(\theta-\phi)}+e^{-i(\theta-\phi)}\right)}\right)$

$= \mathrm{Re}\left(\frac{1-r^2+2ir\sin(\theta-\phi)}{1-2r\cos(\theta-\phi)+r^2}\right)$

$= \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}$ as req.

$\left(\frac{t+z}{t-z}\right) = \left(\frac{1+z/t}{1-z/t}\right) = (1+z/t)(1-z/t)^{-1}$

$= \left(1+\frac{z}{t}\right)\left(1+\frac{z}{t}+\frac{z^2}{t^2}+\frac{z^3}{t^3}+\frac{z^4}{t^4}+\cdots\right)$

$= 1+\frac{2z}{t}+\frac{2z^2}{t^2}+\frac{2z^3}{t^3}+\cdots$ Since $|z|/|t|<1$ series converges by ratio test.

$u(r,\theta) = \frac{1-r^2}{2\pi}\int_0^{2\pi}\frac{u(\phi)\,d\phi}{1-2r\cos(\theta-\phi)+r^2}$

$= \frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\left(\frac{t+z}{t-z}\right)u(\phi)\,d\phi$

$= \frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\left(1+\frac{2z}{t}+\frac{2z^2}{t^2}+\frac{2z^3}{t^3}+\cdots\right)u(\phi)\,d\phi$

Now $z=re^{i\theta}$, $t=e^{i\phi}$

$\therefore u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\left(1+2re^{i(\theta-\phi)}+2r^2e^{2i(\theta-\phi)}+\cdots\right)u(\phi)\,d\phi$

$= \frac{1}{2\pi}\int_0^{2\pi}u(\phi)\,d\phi + \frac{2r}{2\pi}\int_0^{2\pi}\cos(\theta-\phi)\,u(\phi)$

$+ \frac{2r^2}{2\pi}\int_0^{2\pi}\cos 2(\theta-\phi)\,u(\phi)\,d\phi + \cdots + \frac{2r^n}{2\pi}\int_0^{2\pi}\cos n(\theta-\phi)\,u(\phi)\,d\phi$

$+ \cdots$