Question 3 – 25 marks

(i) Using Poisson's formula for a half-plane (Dettman, Exercise 6.3, number 2), construct the harmonic function $u(x, y)$ in the half-plane $y \geq 0$ that gives

$$u(x, y=0) = \begin{cases} 1, & x < 0, \\ 2, & x > 0, \end{cases}$$

$\pi(\cos x + i\sin y)e^{-y}$

showing that

$$u(x, y) = \frac{3}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{x}{y}\right),$$

y $\quad = \dfrac{\pi e^{-y}\cos x}{y}$

with $\tan^{-1}$ taken to lie between $-\pi/2$ and $\pi/2$.

Verify directly that $u(x, y)$ is harmonic and satisfies the boundary conditions as $y \to 0$.

(ii) Express $u(x, y)$ in terms of polar coordinates r, θ where $z = re^{i\theta}$, and hence verify again that $u(r, \theta)$ is harmonic and satisfies the boundary conditions on $y = 0$.

[Note: Laplace's equation in polar coordinates is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

for a function $u(r, \theta)$.]

$\nabla^2 u = \mathrm{div}(\mathrm{grad}(u))$

$\nabla^2 \ln r = \mathrm{div}\left(\frac{1}{r}\right)$

(iii) In the light of Poisson's formula, *deduce* the value of

$$\frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\cos s}{(x-s)^2 + y^2}\, ds.$$

$= \frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{1}{r}\right) = 0$

Question 4 – 25 marks

$x - s = t \quad s = x - t$

Poisson's integral formula for the unit circle (Dettman, p. 238) may be written in the form

$$u(r, \theta) = \frac{1 - r^2}{2\pi}\int_0^{2\pi} \frac{U(\phi)\, d\phi}{1 - 2r\cos(\theta - \phi) + r^2},$$

$\dfrac{\cos x \cos t + \sin x \sin t}{t^2 + y^2}$

for $u(r, \theta) = U(\theta)$ on $r = 1$.

$= \cos x \int \dfrac{\cos t}{t^2 + y^2}$

(i) Show that

$$\frac{1 - r^2}{1 - 2r\cos(\theta - \phi) + r^2} = \mathrm{Re}\left(\frac{t + z}{t - z}\right),$$

where $z = re^{i\theta}$, $t = e^{i\phi}$. By expanding $(t + z)/(t - z)$ for $|z| < |t|$, show that

$$u(r, \theta) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} r^n(a_n \cos n\theta + b_n \sin n\theta),$$

where

$$a_n = \frac{1}{\pi}\int_0^{2\pi} U(\phi)\cos n\phi\, d\phi, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} U(\phi)\sin n\phi\, d\phi.$$

(ii) Hence construct a function $u(r, \theta)$ which is harmonic in the unit circle and takes the boundary value $u = U(\theta) = \cos^2\theta$ on $r = 1$. Express your answer in terms of z.

Explain briefly why your answer can be considered obvious.

[Note:

$$2\cos z_1 \cos z_2 = \cos(z_1 - z_2) + \cos(z_1 + z_2),$$
$$2\sin z_1 \cos z_2 = \sin(z_1 - z_2) + \sin(z_1 + z_2).]$$