5(iii) Comparing

$$\frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\cos s}{(x-s)^2+y^2}\, ds$$

with Poisson's formula

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y\, G(s)}{y^2+(x-s)^2}\, ds$$

we see that $G(s) = \cos s$.

So we require a harmonic fn. $u(x,y)$ that satisfies

$$u(x,y) = \cos x \quad \text{on} \quad y=0$$

i.e. we require an analytic fn. $f(z)$ (with real part $u(x,y)$) that gives e^{ix} on $y=0$ (e^{ix} has real part $\cos x$)

This suggests we take

$$f(z) = e^{iz}$$

which gives real part $e^{-y}\cos x$ (and this $= \cos x$ on $y=0$).

The function $u(x,y)$ must also be bounded as $y \to +\infty$; $e^{-y}\cos x$ is bounded. Hence

$$\frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\cos s}{(x-s)^2+y^2}\, ds = e^{-y}\cos x.$$

$\frac{5}{25}$

Notes (i) Other possible functions such as e^{-iz} or $\cos z$ can be excluded because, although analytic and giving the right behaviour on $y=0$ (namely e^{-ix}, with real part $\cos x$ for e^{-iz} or $\cos x$ for $\cos z$)

they are not bounded as $y \to +\infty$.

(ii) $\left| \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\cos s}{(x-s)^2+y^2}\, ds \right| \le \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{ds}{(x-s)^2+y^2}$ since $|\cos s| \le 1$ and $y > 0$.

$= \frac{y}{\pi} \cdot \frac{\pi}{y}$ evidi. integral

$= 1$ i.e. $\left|\frac{y}{\pi}\int_{-\infty}^{\infty} \text{—}\right| \le 1.$