Please make sure that the assignment number is correctly entered on your PT3 form as

$\boxed{\text{M828 01}}$.

Question 1 – 25 marks

(i) Show that the function

$$f(w) = \frac{1-w}{1+w}$$

is purely imaginary on the circle $w = e^{i\theta}$.

(ii) Let D be the domain $\{z : |z| < 1,\ 0 < \arg(z) < \pi/3\}$. Show that $w = z^3$ maps D to a semi-circular region D' in the w-plane. Determine the values of $\arg f(w)$, the argument of $f(w)$, on the boundary of D'.

(iii) Hence construct a function which is harmonic in D and takes the value 2 on the curved arc in the boundary of D and the value 1 on the two line segments in the boundary of D.

Question 2 – 25 marks

(i) An isosceles triangle ABC with vertex angles $\pi/4$, $\pi/4$ and $\pi/2$ has vertices at the points $(u, v) = A(-a, 0)$, $B(a, 0)$ and $C(0, a)$ in the w-plane, where $w = u + iv$. Determine the Schwarz–Christoffel transformation that maps the triangle to the upper half-plane $\mathrm{Im}(z) > 0$, with A going to $z = -1$, B going to $z = 1$, and C going to infinity. Show that

$$w = \frac{a}{I}\int_0^z (1-t^2)^{-3/4}\,dt,$$

where the constant I is the integral

$$I = \int_0^1 (1-t^2)^{-3/4}\,dt.$$

(ii) The integral I may be evaluated in terms of the *beta function* $B(z, w)$ and the *gamma function* $\Gamma(z)$, defined by (see also Dettman, Section 4.10)

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\,dt,$$

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt,$$

where $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(w) > 0$, and related by

$$B(z,w) = \frac{\Gamma(z)\,\Gamma(w)}{\Gamma(z+w)}.$$

Show that

$$\int_0^1 \frac{dt}{(1-t^2)^{3/4}} = \frac{\Gamma^2(\frac{1}{4})}{2(2\pi)^{1/2}}.$$

[Note: The gamma function is important and arises elsewhere in this course. It satisfies the relation

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z},$$

and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Handbooks of mathematical functions give $\Gamma(\frac{1}{4}) = 3.6256$.]