

Question 4 - 25 marks

This question is concerned with the method of stationary phase.

- (i) Give a sketch (using a computer if possible) of the functions

$$h(t) \equiv t^3 - t, \quad \cos h(t), \quad \cos 2h(t), \quad \cos 10h(t)$$

in the domain  $0 \leq t < \infty$ , and hence identify the point  $t = t_0$  of stationary phase for the integral

$$f(x) = \int_0^{\infty} \cos(x(t^3 - t)) dt$$

as  $x \rightarrow \infty$  through real values.

Following the procedure in Dettman (pp. 457-8), or otherwise, derive the result

$$f(x) \sim \left(\frac{\pi}{3xt_0}\right)^{1/2} \cos\left(x(t_0^3 - t_0) + \frac{\pi}{4}\right), \quad \text{as } x \rightarrow +\infty.$$

[Note: There is a typographical error in Dettman's concluding asymptotic formula on p. 458:  $h'(a)$  should be replaced by  $h''(a)$ .]

- (ii) In the Course Notes (Section 5.3.6, Example 12) it is shown that

$$\int_0^{\infty} e^{it^2} dt = \frac{1}{2}\sqrt{\pi}e^{i\pi/4}.$$

Considering the integral

$$\int_C z^\alpha e^{i\beta z} dz,$$

where the closed contour  $C$  consists of the real axis from  $z = 0$  to  $z = R$ , plus the quarter-circle  $z = Re^{i\theta}$ ,  $0 \leq \theta \leq \pi/2$ , plus the imaginary axis from  $z = iR$  to  $z = 0$ , show in a similar manner that

$$\int_0^{\infty} t^\alpha e^{i\beta t} dt = \frac{\Gamma(1+\alpha)}{\beta^{1+\alpha}} e^{i(1+\alpha)\pi/2} \quad (\beta > 0).$$

What inequality constraint on the real constant  $\alpha$  is needed for this result to be valid?

[You may use the integral formula for the gamma function quoted in TMA 01.]

- (iii) Use the method of stationary phase to obtain the leading-order behaviour of the integral

$$\int_0^1 \tan t \cos(xt^4) dt$$

as  $x \rightarrow +\infty$ .

Now that you have completed your assignments for M828, please complete the course evaluation sheet which was sent to you with this *Assignment Booklet*. Please return it to the Courses Office, Faculty of Mathematics and Computing, as directed. Thank you.