M828

## Comments on TMA 03

This tma was done quite well, though I suspect that many of you struggled long and hard with certain parts of the questions. If this was a constructive struggle, from which you will have gained much, then that is all to the good! Also, I have a feeling many of you enjoyed doing some Residue Theory! This TMA may have been a bit on the long side - as I kept extended it whenever I got ideas for a question. Apologies for that!

Specific comments:

**Qu. 1. (i)** I enjoyed constructing this question as an illustration of the Riemann-Lebesgue lemma [Dettman, Thm 8.1.1], but probably I have made it too straightforward! Perhaps this is an example of the setter gaining more from the exercise than the students who then tackle the question, unaware of the many steps in setting the question that took place behind the scenes. But I wonder what would have been the result if I had omitted the guidance of the first trigonometrical formula.

**(ii)** The initial value theorem quoted in the Hint is not in Dettman, so you may want to suitably annotate your copy. The determination of the constant $C$ is (too?) easy, given my Hint.

**(iii)**
My idea in setting this part of the question was to make use of the earlier Laplace transform ideas, as was obvious from the context of the question. There arises the question of the existence of the quoted integrals, and it is important that this is given some consideration even if the question does not specifically ask for this. I tried to give a hint on this, with the "if appropriate" wording, but I didn't want to be too specific as then there would have been no challenge in the question.

The integrals

$$I_1 = \int_0^\infty J_0(t)dt \quad \text{and} \quad I_2 = \int_0^\infty t^2 J_0(t)dt$$

may be thought of as following from the Laplace integral

$$I(s) = \int_0^\infty e^{-st} J_0(t)dt, \quad s > 0.$$

By taking the limit $s \to 0_+$ (the Laplace transform integral requires $s > 0$) in the result $I = 1/(1+s^2)^{1/2}$, we have $I \to 1$ as $s \to 0_+$, suggesting that

$$I_1 = \int_0^\infty J_0(t)dt = 1.$$

Similarly, twice differentiating $I(s)$ with respect to $s$ and then letting $s \to 0_+$ suggests that

$$I_2 = \int_0^\infty t^2 J_0(t)dt = -1.$$

But do these improper integrals actually exist (ie, are they convergent)? The obvious worry in $I_2$ is the behaviour of the integrand for large $t$, because of the $t^2$-term (making the integrand grow with $t$, albeit in an oscillatory manner), suggesting perhaps that $I_2$ does *not* exist.

It is instructive to make up a more straightforward case, where the integration is easy. So consider the integral

$$K(s) = \int_0^\infty e^{-st} \cos t dt = \frac{s}{1+s^2}, \quad s > 0.$$