TMA M828 03 **Cut-off date** 18 June 2003

Please make sure that the assignment number is correctly entered on your PT3 form as

$\boxed{\text{M828 03}}$.

Question 1 – 25 marks

(i) Show that

$$\frac{\sin(2m+1)\theta}{\sin\theta} = 1 + 2\cos 2\theta + 2\cos 4\theta + \cdots + 2\cos 2m\theta,$$

for integer m, and hence that

$$\int_0^{\pi/2} \frac{\sin(2m+1)\theta}{\sin\theta}\,d\theta = \frac{\pi}{2}.$$

Why is this result not in contradiction to the Riemann–Lebesgue lemma (Theorem 8.1.1 of Dettman, p. 350)?

Apply the Riemann–Lebesgue lemma to

$$\int_{\pi/4}^{\pi/2} \frac{\sin(2m+1)\theta}{\sin\theta}\,d\theta,$$

and hence obtain a series expansion for π.

(ii) The differential equation

$$t\frac{d^2x}{dt^2} + \frac{dx}{dt} + tx = 0,$$

subject to $x(0) = 1$, has solution $x(t) = J_0(t)$ for the Bessel function $J_0(t)$. By considering the Laplace transform L of the differential equation, deduce that

$$L\left[J_0(t)\right] = \frac{C}{(1+s^2)^{1/2}}.$$

Determine the constant C.

[Hint: Apply the initial value theorem for Laplace transforms, namely

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s).]$$

(iii) Evaluate (if appropriate) the integrals

$$\int_0^\infty J_0(t)\,dt \quad \text{and} \quad \int_0^\infty t^2 J_0(t)\,dt.$$

$J = \sum a_n t^n$