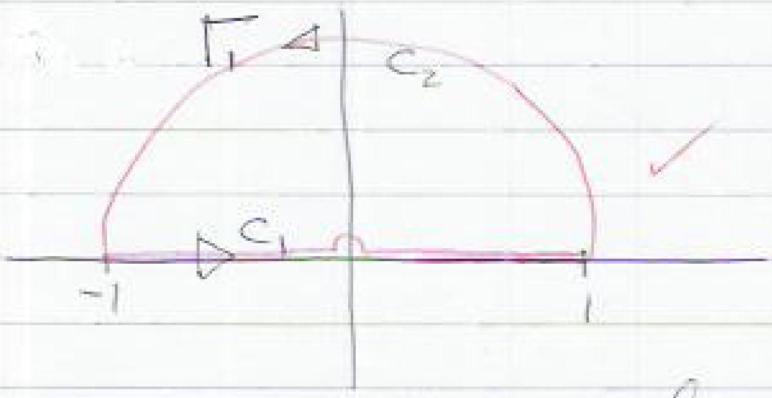
$\frac{(-1)^n}{i}\frac{1}{(2i)^{2n}}\frac{(2n)!}{(n!)^2}$ where from?

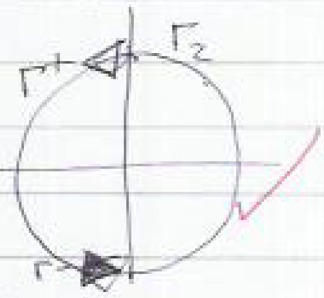


There are no poles of the integrand in Γ_1

$$\int_{\Gamma}\frac{(-1)^n}{(2i)^{2n}}\left(z-\frac{1}{z}\right)^{2n}\frac{dz}{iz}=0$$

$$\therefore \int_{C_1} = -\int_{C_2} \Rightarrow \int_{C_2} = \int_{-C_1} = \int_{1}^{-1}$$

But also consider Γ_2:



There is a pole of the integrand inside Γ_2 with residue $\frac{(-1)^n(2n)!}{(2i)^{2n}(n!)^2\, i}$

Not completely clear.

$$\therefore \int_{\Gamma_2}\frac{(-1)^n}{(2i)^{2n}}\left(z-\frac{1}{z}\right)^{2n}\frac{dz}{iz} = 2\pi i\frac{(-1)^n(2n)!}{(2i)^{2n}(n!)^2\, i}$$

$$= \frac{(-1)^n(2n)!}{(2i)^{2n}(n!)^2}\frac{2\pi i}{i} = \frac{(2n)!}{2^{2n}(n!)^2}2\pi$$

But $\int_{\Gamma_+} = +\int_{\Gamma_-}$ since in $\mathbb{C}-\{0\}$

Both contours have same start point and end point, and integrand is differentiable in $\mathbb{C}-\{0\}$

$$\therefore \int_{\Gamma_+} = \int_{\Gamma_-} = \frac{(2n)!}{2^{2n}(n!)^2}\pi$$