Question 2 – 25 marks

Obtain the Laplace transform of $J_0(x)$ as follows (see Question 1(ii)).

Let

$$a_n = \int_0^{\pi} (\sin\theta)^{2n}\, d\theta.$$

Show that

$$a_n = \frac{(2n)!}{2^{2n}(n!)^2}\pi.$$

[Note: You may wish to do this using the Residue theorem applied to the integral

$$\oint_C \left(z - \frac{1}{z}\right)^{2n} \frac{1}{z}\, dz,$$

with C the unit circle and $z = e^{i\theta}$; the binomial expansion may prove useful.]

Noting that

$$\cos t = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{2n}, \quad J_0(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \frac{t^{2n}}{2^{2n}},$$

show that

$$J_0(t) = \frac{1}{\pi} \int_0^{\pi} \cos(t \sin\theta)\, d\theta.$$

Hence deduce that

$$L[J_0(t)] = \frac{s}{\pi} \int_0^{\pi} \frac{d\theta}{s^2 + \sin^2\theta}, \quad s > 0.$$

Finally, using complex analysis (or otherwise) to evaluate the integral, deduce the Laplace transform of $J_0(t)$.

Question 3 – 25 marks

Consider the initial value problem

$$\frac{d^2y}{dt^2} + 4y = f(t), \quad y(t=0) = 3, \quad \frac{dy}{dt}(t=0) = -1,$$

for an arbitrary function $f(t)$. Use the Laplace transform method to show that $y = y(t)$ is of the form

$$y = a\cos 2t + b\sin 2t + \tfrac{1}{2}\int_0^t f(\tau)\sin 2(t-\tau)\, d\tau$$

for constants a and b.

Verify directly that this solution satisfies the differential equation and initial conditions.

Deduce (without performing the integration directly) the value of the integral

$$\int_0^t \tau \sin 2(t-\tau)\, d\tau.$$

[Note: For functions a, b of parameter α, we have

$$\frac{d}{d\alpha}\left\{\int_a^b f(s,\alpha)\, ds\right\} = f(b,\alpha)\frac{db}{d\alpha} - f(a,\alpha)\frac{da}{d\alpha} + \int_a^b \frac{\partial f}{\partial \alpha}\, ds.]$$