**Question 4** – 25 marks

(i) Use the Laplace transform to solve the Abel integral equation

$$\int_0^t \frac{y(s)}{(t-s)^{1/2}}\,ds = t + t^2, \quad t > 0.$$

(ii) Verify your solution by substitution back into the integral equation.

(iii) Show, in a similar manner, that

$$\int_0^t \frac{y(s)}{(t-s)^{1/2}}\,ds = \int_0^t \phi(s)\,ds$$

has solution

$$y(t) = \frac{1}{\pi}\int_0^t \frac{\phi(s)}{(t-s)^{1/2}}\,ds.$$

Verify that this recovers the solution obtained in part (i).

[Note: The following results may prove useful. For the Laplace transform $L$,

$$L(t^\alpha) = \Gamma(1+\alpha)s^{-(1+\alpha)}, \quad 1+\alpha > 0,$$

$$\Gamma(1+z) = z\Gamma(z), \quad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}, \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},$$

$$\int_0^1 t^{z-1}(1-t)^{w-1}\,dt = \frac{\Gamma(z)\,\Gamma(w)}{\Gamma(z+w)}.]$$

---

$$1 \int \frac{2s^{1/2}}{\pi(t-s)^{1/2}} + \frac{8}{3\pi}\frac{s^{3/2}}{(t-s)^{1/2}}$$

$$= \frac{2}{\pi}\int \sqrt{\frac{t\sin^2\theta}{t\cos^2\theta}}\cdot 2t\sin\theta\cos\theta + \frac{8}{3\pi}\frac{t^{3/2}\sin^3\theta\; 2t\sin\theta\cos\theta}{\sqrt{t}\cos\theta}$$

$$= \frac{2}{\pi}\int 2t\sin^2\theta + \frac{16}{3\pi}\int t^2\sin^4\theta\,d\theta$$

$$= \frac{4t}{\pi}\int \sin^2\theta \quad + \cancel{\frac{2\cdot 8\, t^2}{3\pi}} \cdot \cancel{\frac{1\cdot 3}{2\cdot 4}}\,\frac{\pi}{2}$$