(This is evidently similar to I in that cos is similar to J_0, though with the perhaps significant difference that $J_0(t) \to 0$ as $t \to \infty$ whereas $\cos t$ has no limit for $t \to \infty$.) In the limit $s \to 0_+$ we have $K = 0$, but in fact the integral

$$\int_0^\infty \cos t dt$$

does not exist. The integral $\int_0^T \cos t dt$, for finite upper limit T, is $\sin T$, but $\lim \sin T$ as $T \to \infty$ does not exist. So there are problems with integrals of this form and we must be careful.

The Bessel function $J_0(t)$ declines to zero as $t \to \infty$ roughly like $1/t^{1/2}$, oscillating at the same time [properties of Bessel functions are covered by Dettman]. But the integral involving t^2 clearly becomes large (though oscillating) for large t, suggesting that the integral diverges. A fuller investigation of this is needed. I did not expect a very detailed investigation of these issues but I felt that you should have indicated an awareness of the problem, and sounded a note of caution. Accordingly, I allocated only a couple of marks to this aspect, giving 3 marks for "showing" that $I_1 = 1$ and $I_2 = -1$, and 2 marks for appropriate words of caution. My own considerations suggested that the result $\int_0^\infty J_0(t)dt = 1$ is correct but the integral $\int_0^\infty t^2 J_0(t) dt$ *diverges* [so the result $I_2 = -1$ is wrong].

Qu. 2.

This question gave you a nice chance to refresh your ideas about residues and contour integrals, all no doubt at your finger tips from previous courses! One point worth noticing is that it is often useful to make use of L'Hospital's rule in working out residues. For example, my solution for the last part of the question leads to the contour integral

$$\int_C \frac{z}{(z^2-1)^2 - 4s^2z^2} dz$$

where C is the unit circle. There are four simple poles, two inside C and two outside. Let z_0 denote any of these simple poles, and so z_0 satisfies $(z^2-1)^2 - 4s^2z^2 = 0$. Then the residue at $z = z_0$ is

$$\begin{aligned} \lim_{z \to z_0} \frac{z(z-z_0)}{(z^2-1)^2 - 4s^2z^2} &= z_0 \lim_{z \to z_0} \frac{(z-z_0)}{(z^2-1)^2 - 4s^2z^2} \\ &= \frac{1}{4(z_0^2 - 1 - 2s^2)} \end{aligned}$$

where we have used L'Hospital's rule in the above. The residue can then be easily calculated for the specific poles in question (ie, for each z_0, as appropriate).

Qu. 3.

This question was in the main well done. My rider at the end of the question, asking you deduce the value of an integral, did not trouble most of you. You realised that it requires $f(t) = t$, for which the original differential equation has (by inspection) a simple general solution [a particular integral is $t/4$]. This can then be related to the given integral, and its value deduced.

Qu. 4. In general Qu. 4 was well done. But an issue arises re. the process of *verification*. My purpose in asking for a verification was to get you to carry out a check on your working. In the real world one always has to carry out many checks; it is all too easy to make errors and so it is important to guard against this by thinking up various checks. These should be as *independent* as possible of the procedure used to obtain the result/solution to be checked; otherwise there is a danger of the same mistake being made twice and so passing undetected! Here in carrying out a verification some of you used large parts of the earlier working (perhaps directly, using the same Laplace transform formula) and so failed to supply an independent verification. (Because it was only a checking procedure, I have been lenient in my marking but the point still stands.) Finally, it is worth noting that even if a *derivation* is suspect, the actual *result* may be perfectly correct, as a verification or check may support.

A direct verification amounts to showing that

$$\frac{2}{\pi} \int_0^t (s^{1/2} + \frac{4}{3}s^{3/2})(t-s)^{-1/2} ds = t^2 + t^2.$$