(ii) Assuming the result

$$P'_{n+1}(z) - P'_{n-1}(z) = (2n+1)P_n(z),$$

show that

$$\int_{-1}^{1} z^n P_n(z)\,dz = \left(\frac{n}{2n+1}\right)\int_{-1}^{1} z^{n-1} P_{n-1}(z)\,dz.$$

Hence deduce that

$$\int_{-1}^{1} z^n P_n(z)\,dz = \frac{2^{n+1}(n!)^2}{(2n+1)!}.$$

Question 4 – 25 marks

(i) Let $J_n(z)$ denote the Bessel function of order n. Set

$$\phi(z,t) = \sum_{n=-\infty}^{\infty} t^n J_n(z),$$

and assume that the series converges. Obtain a first-order differential equation relating $\partial\phi/\partial z$ and $\phi(z,t)$, and hence deduce that

$$\phi(z,t) = \exp\left[\tfrac{1}{2}z\left(t - \frac{1}{t}\right)\right].$$

(ii) By setting $t = e^{i\theta}$ and also $t = -e^{i\theta}$, or otherwise, show that for *complex* z,

$$\sin(z\sin\theta) = 2\sum_{n=0}^{\infty} J_{2n+1}(z)\sin(2n+1)\theta,$$

and obtain a similar relation for $\cos(z\sin\theta)$.

(iii) Deduce that

$$\tfrac{1}{2}z = J_1(z) + 3J_3(z) + 5J_5(z) + 7J_7(z) + \cdots.$$

[Note: The following properties of Bessel functions may be assumed:

$$2J'_n(z) = J_{n-1}(z) - J_{n+1}(z),$$
$$J_n(-z) = (-1)^n J_n(z),$$
$$J_{-n}(z) = (-1)^n J_n(z),$$
$$J_n(0) = 0 \quad (n > 0),$$
$$J_0(0) = 1.]$$