

Please make sure that the assignment number is correctly entered on your PT3 form as

M828 02

Question 1 - 25 marks

Show that the differential equation

$$x^2 y'' + 3xy' + (1+x)y = 0$$

has a regular singular point at $x = 0$. Use the method of Frobenius in the form

$$y = y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\alpha}, \quad a_0 \neq 0,$$

to obtain a solution of the differential equation.

Determine α , and the explicit form of the coefficients a_n given that $a_0 = 1$. What is the radius of convergence of the expansion?

What form would a second, independent, series solution $y = y_2(x)$ about $x = 0$ take? [You are not asked to obtain this second solution.]

Question 2 - 25 marks

Show that the ordinary differential equation

$$z^3 w'' + zw' - w = 0$$

has an irregular singular point at $z = 0$. By considering a solution of the Frobenius

form $w = \sum_{n=0}^{\infty} a_n z^{n+\alpha}$, show that only one value of α is possible, and thus find a non-zero solution of the differential equation.

Hence obtain the general solution of the differential equation. What solution $w = w(z)$ satisfies the two boundary conditions $w(z = 1) = 1$ and $w(z = 2) = 2$?

Question 3 - 25 marks

(i) Starting from Legendre's equation

$$\frac{d}{dz} \left((1-z^2) \frac{dP_n}{dz} \right) + n(n+1)P_n(z) = 0,$$

show that

$$n(n+1) \int_{-1}^1 P_m(z) P_n(z) dz = \int_{-1}^1 (1-z^2) P'_m(z) P'_n(z) dz, \quad = (1-z^2) P'_m P_n$$

where $P'_n(z) = dP_n(z)/dz$. Hence deduce that

$$\int_{-1}^1 P_m(z) P_n(z) dz = 0, \quad n \neq m.$$

Deduce also that

$$\int_{-1}^1 z^k P_n(z) dz = 0$$

when $k = 0, 1, 2, \dots, n-1$.

$$- \int (P_n (1-z^2))' P_n dz$$