

$$4) V_1 = \frac{\partial}{\partial x^1} = \frac{\partial x^1}{\partial \xi^1} \frac{\partial}{\partial x^1} + \frac{\partial x^2}{\partial \xi^1} \frac{\partial}{\partial x^2} + \frac{\partial x^3}{\partial \xi^1} \frac{\partial}{\partial x^3}$$

$$= \left(1 + \left(\frac{\xi^1}{\xi^2}\right)^2 - \left(\frac{\xi^2}{\xi^3}\right)^2\right) \frac{\partial}{\partial x^1} + 2\frac{\xi^1 \xi^2}{\xi^3} \frac{\partial}{\partial x^2} + 2\frac{\xi^1}{\xi^3} \frac{\partial}{\partial x^3}$$

This may also be written

$$V_1 = \left(1 - \left(\frac{\xi^1}{\xi^2}\right)^2 - \left(\frac{\xi^2}{\xi^3}\right)^2, -2\frac{\xi^1 \xi^2}{\xi^3}, 2\frac{\xi^1}{\xi^3}\right)$$

$$V_2 = \frac{\partial}{\partial \xi^2} = \frac{\partial x^1}{\partial \xi^2} \frac{\partial}{\partial x^1} + \frac{\partial x^2}{\partial \xi^2} \frac{\partial}{\partial x^2} + \frac{\partial x^3}{\partial \xi^2} \frac{\partial}{\partial x^3}$$

$$= \frac{\xi^1 \xi^2}{\xi^3} \frac{\partial}{\partial x^1} - \left(1 + \left(\frac{\xi^1}{\xi^2}\right)^2 - \left(\frac{\xi^2}{\xi^3}\right)^2\right) \frac{\partial}{\partial x^2} + 2 \frac{\xi^2}{\xi^3} \frac{\partial}{\partial x^3}$$

and this may be written

$$V_2 = \left(2\frac{\xi^1 \xi^2}{\xi^3}, -\left(1 + \left(\frac{\xi^1}{\xi^2}\right)^2 - \left(\frac{\xi^2}{\xi^3}\right)^2\right), -2\frac{\xi^2}{\xi^3}\right)$$

$$V_1 \times V_2 = \det \begin{bmatrix} 1 - \left(\frac{\xi^1}{\xi^2}\right)^2 - \left(\frac{\xi^2}{\xi^3}\right)^2 & -2\frac{\xi^1 \xi^2}{\xi^3} & 2\frac{\xi^1}{\xi^3} \\ 2\frac{\xi^1 \xi^2}{\xi^3} & -\left(1 + \left(\frac{\xi^1}{\xi^2}\right)^2 - \left(\frac{\xi^2}{\xi^3}\right)^2\right) & -2\frac{\xi^2}{\xi^3} \\ i & j & k \end{bmatrix}$$

$$= i \left(4\frac{\xi^1}{\xi^3} \left(\frac{\xi^2}{\xi^3}\right)^2 + 2\frac{\xi^1}{\xi^3} \left(1 + \left(\frac{\xi^1}{\xi^2}\right)^2 - \left(\frac{\xi^2}{\xi^3}\right)^2\right)\right)$$

$$= -j \left(\left(1 - \left(\frac{\xi^1}{\xi^2}\right)^2 - \left(\frac{\xi^2}{\xi^3}\right)^2\right) \left(-2\frac{\xi^2}{\xi^3}\right) + 4\left(\frac{\xi^1}{\xi^2}\right)^2 \frac{\xi^2}{\xi^3}\right)$$

$$+ k \left(-\left(1 - \left(\frac{\xi^1}{\xi^2}\right)^2 - \left(\frac{\xi^2}{\xi^3}\right)^2\right) \left(2\frac{\xi^1 \xi^2}{\xi^3}\right) + \left(1 + \left(\frac{\xi^1}{\xi^2}\right)^2 - \left(\frac{\xi^2}{\xi^3}\right)^2\right) \left(2\frac{\xi^1}{\xi^3}\right)\right)$$

$$= \left(2\frac{\xi^1}{\xi^3} \left(1 + \left(\frac{\xi^1}{\xi^2}\right)^2 + \left(\frac{\xi^2}{\xi^3}\right)^2\right), 2\frac{\xi^2}{\xi^3} \left(1 + \left(\frac{\xi^1}{\xi^2}\right)^2 + \left(\frac{\xi^2}{\xi^3}\right)^2\right), \right.$$

$$\left. + \left(4\left(\frac{\xi^1}{\xi^2}\right)^2 \left(\frac{\xi^2}{\xi^3}\right)^2 - \left(1 - \left(\frac{\xi^1}{\xi^2}\right)^2 - \left(\frac{\xi^2}{\xi^3}\right)^2\right) \right) k\right)$$

$$= 2\frac{\xi^1}{\xi^3} \left(1 + \left(\frac{\xi^1}{\xi^2}\right)^2 + \left(\frac{\xi^2}{\xi^3}\right)^2\right) i + 2\frac{\xi^2}{\xi^3} \left(1 + \left(\frac{\xi^1}{\xi^2}\right)^2 + \left(\frac{\xi^2}{\xi^3}\right)^2\right) j$$

$$+ \left(4\left(\frac{\xi^1}{\xi^2}\right)^2 \left(\frac{\xi^2}{\xi^3}\right)^2 - \left(1 - \left(\frac{\xi^1}{\xi^2}\right)^2 - \left(\frac{\xi^2}{\xi^3}\right)^2\right) \right) k$$

$$= 2\frac{\xi^1}{\xi^3} \left(1 + \left(\frac{\xi^1}{\xi^2}\right)^2 + \left(\frac{\xi^2}{\xi^3}\right)^2\right) i + 2\frac{\xi^2}{\xi^3} \left(1 + \left(\frac{\xi^1}{\xi^2}\right)^2 + \left(\frac{\xi^2}{\xi^3}\right)^2\right) j$$

$$+ \left(4\left(\frac{\xi^1}{\xi^2}\right)^2 \left(\frac{\xi^2}{\xi^3}\right)^2 - \left(1 - \left(\frac{\xi^1}{\xi^2}\right)^2 - \left(\frac{\xi^2}{\xi^3}\right)^2\right) \right) k$$

$$= 2\frac{\xi^1}{\xi^3} \left(1 + \left(\frac{\xi^1}{\xi^2}\right)^2 + \left(\frac{\xi^2}{\xi^3}\right)^2\right) i + 2\frac{\xi^2}{\xi^3} \left(1 + \left(\frac{\xi^1}{\xi^2}\right)^2 + \left(\frac{\xi^2}{\xi^3}\right)^2\right) j$$

$$+ \left(\left(\frac{\xi^1}{\xi^2}\right)^2 + \left(\frac{\xi^2}{\xi^3}\right)^2 - 1\right) \left(\left(\frac{\xi^1}{\xi^2}\right)^2 + \left(\frac{\xi^2}{\xi^3}\right)^2 + 1\right) k$$

$$\text{while } |V_1| \cdot |V_2| = \left(1 - \left(\frac{\xi^1}{\xi^2}\right)^2 - \left(\frac{\xi^2}{\xi^3}\right)^2\right)^2 + 4\left(\frac{\xi^1}{\xi^2}\right)^2 \left(\frac{\xi^2}{\xi^3}\right)^2 + 4\left(\frac{\xi^1}{\xi^2}\right)^2 \left(\frac{\xi^2}{\xi^3}\right)^2$$

$$\times \left(4\left(\frac{\xi^1}{\xi^2}\right)^2 \left(\frac{\xi^2}{\xi^3}\right)^2 + \left(1 - \left(\frac{\xi^1}{\xi^2}\right)^2 - \left(\frac{\xi^2}{\xi^3}\right)^2\right)^2 + 4\left(\frac{\xi^1}{\xi^2}\right)^2 \left(\frac{\xi^2}{\xi^3}\right)^2\right)^{1/2}$$