Question 2

(i) To show you have a 1-parameter group, compute $\rho_s(\rho_t(x))$:

$$\rho_s(\rho_t(x)) = \rho_s(y + R_t(x-y) + t\mathbf{v}) = y + R_s((y + R_t(x-y) + t\mathbf{v}) - y) + s\mathbf{v} =$$

$$y + R_s(R_t(x-y)) + R_s(t\mathbf{v}) + s\mathbf{v} = y + R_{s+t}(x-y) + (s+t)\mathbf{v} = \rho_{s+t}(x).$$

To show the elements are isometries, observe that they are composites of translations and rotations, which are isometries.

(ii) $\lambda = 0$ implies $\rho_t(y+v) = y + R_t(v)$, so ρ_t is a rotation about y. Since $R_t(n) = n$, the axis of rotation is the line through y in the direction of n.

If $v = 0$ ρ_t is a translation by $t(\lambda n)$.

In the general case the orbits are helices around the axis.

(iii) We need to calculate $V = \frac{d}{dt}(\rho_t)\Big|_{t=0}$, and express the answer in the form $\xi^a X_a + \tau^a T_a$.

The tricky bit is $\frac{d}{dt}(R_t)_{t=0} = (w^a_b) = \begin{pmatrix} 0 & -\nu n^3 & \nu n^2 \\ \nu n^3 & 0 & -\nu n^1 \\ -\nu n^2 & \nu n^1 & 0 \end{pmatrix}$. The entries in the matrix are explained on p. 199. Then V, the Killing vector field corresponding to ρ_t, is given by

$$V^a = w^a_b(x^b - y^b) + \lambda n^a = (\lambda n^a - w^a_b y^b) + w^a_b x^b.$$

So $V = (\lambda n^a - w^a_b y^b)T_a + \nu n^a X_a$.

The coefficients are: $\tau^a = (\lambda n^a - w^a_b y^b)$, $\xi^a = \nu n^a$.

(iv) Now if we are given the ξ^a and the τ^a, and have to find λ, ν, $\mathbf{y}$ and $\mathbf{n}$. Because the vector $\mathbf{n}$ is a unit vector, we can solve the equations $\xi^a = \nu n^a$; in vector notation, $\mathbf{n} = \xi/\|\xi\|$ and $\nu = \|\xi\|$. To solve the equations $\tau^a = (\lambda n^a - w^a_b y^b)$, use the fact that $w^a_b y^b = \upsilon\mathbf{n}\times\mathbf{y}$, so, in vector notation, $\boldsymbol{\tau} = \lambda\mathbf{n} - \nu\mathbf{n}\times\mathbf{y}$. This implies that $\boldsymbol{\tau} = \lambda\xi/\|\xi\| - \nu\xi/\|\xi\|\times\mathbf{y}$, so $\boldsymbol{\tau}.\xi = \lambda$, so $\boldsymbol{\tau} = \lambda\xi/\|\xi\| - \nu\xi/\|\xi\|\times\mathbf{y}$. It follows that $\mathbf{y}$ is a multiple of $\boldsymbol{\tau}\times\mathbf{n}$ and is only defined up to a multiple of $\mathbf{n}$. This is OK, because y only fixes a point on the axis, but any point on the axis will do.

Question 3

(i) Since $\langle U_a, \theta^b\rangle = \delta^b_a$, and $L_V\langle W, \alpha\rangle = V(\langle W, \alpha\rangle) = \langle [V, W], \alpha\rangle + \langle W, L_V\alpha\rangle$, a calculation of $V(\langle U_a, \theta^b\rangle) = 0$ shows that $\lambda^b_a + \mu^b_a = 0$, so $\lambda^b_a = -\mu^b_a$.