Question 2 (20 marks)

A one-parameter family ρ_t of affine transformations of $\mathcal{E}^3$ is defined as follows. Let y be a fixed point of $\mathcal{E}^3$, R_t a one-parameter group of rotation matrices whose angular speed is ν and whose axis is specified by a fixed unit vector $\mathbf{n}$, and $\mathbf{v}$ a fixed vector which is a scalar multiple of $\mathbf{n}$. Then for any $x \in \mathcal{E}^3$ we set

$$\rho_t(x) = y + R_t(x - y) + t\mathbf{v}.$$

The first three parts of the question are concerned with the properties of this one-parameter family of transformations of $\mathcal{E}^3$.

(i) Show that ρ_t is a one-parameter group of isometries of $\mathcal{E}^3$. [5]

(ii) Show that if $\mathbf{v}$ is the zero vector and $\nu \neq 0$, then ρ_t consists of rotations about the line through y in the direction of $\mathbf{n}$. Describe the nature of the transformations defined by ρ_t when $\nu = 0$ (so that R_t degenerates to the constant one-parameter group consisting of the identity alone) but $\mathbf{v}$ is non-zero.

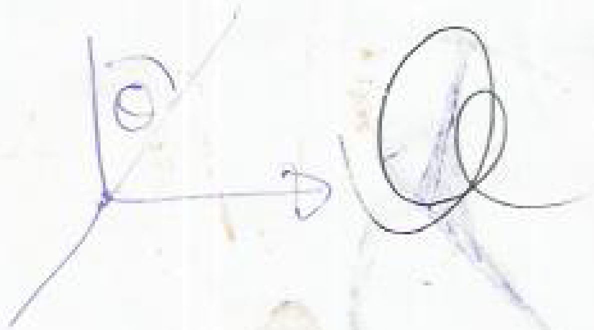
Show that if, on the other hand, both ν and $\mathbf{v}$ are non-zero then the line through y in the direction of $\mathbf{n}$, while fixed as a whole under ρ_t, is not pointwise fixed. Describe briefly the orbits of points not on the line in this case. [5]

An arbitrary system of orthogonal coordinates is taken for $\mathcal{E}^3$, i.e. one in which y is *not* necessarily the origin nor $\mathbf{n}$ necessarily a coordinate axis.

(iii) Find the generator of the one-parameter group ρ_t, i.e. the corresponding Killing field, expressing your answer as a linear combination of the vector fields X_a and T_a, $a = 1, 2, 3$, which are the generators of rotations about, and of translations along, the coordinate axes. [6]

The purpose of the final part of the question is to show that ρ_t is the most general kind of infinitesimal isometry of $\mathcal{E}^3$, by showing that every Killing field on $\mathcal{E}^3$ — in other words, every linear combination of the vector fields X_a and T_a, $a = 1, 2, 3$ — is the infinitesimal generator of some ρ_t.

(iv) Show that every vector field on $\mathcal{E}^3$ of the form $\xi^a X_a + \tau^a T_a$, where the ξ^a and τ^a are constants, may be obtained by starting with a suitable ρ_t (i.e. with suitable y, $\mathbf{n}$, ν and $\mathbf{v}$, which should be found in terms of ξ^a and τ^a). [4]



$(x-y)_\perp \cdot (x-y)_\parallel \cdot (R_t(x-y))_\perp \cdot (R_t(x-y))_\parallel$