(16)

iii) $L_X g = 0$ for X a Killing field.

$\Rightarrow X g(U_a, U_b) - g([X, U_a], U_b) - g(U_a, [X, U_b]) = 0$

$\Rightarrow X(\delta_{ab}) - g(\lambda_a^c U_c, U_b) - g(U_a, \lambda_b^c U_c) = 0$

$0 = \lambda_a^c g(U_c, U_b) + \lambda_b^c g(U_a, U_c)$

ie $0 = \lambda_a^c \delta_{cb} + \lambda_b^c \delta_{ac}$

So the matrix with entries λ_a^c is skew symmetric

[When $b=c$ in 1st term

$\lambda_a^b + \lambda_b^c \delta_{ac} = 0$

$b=a$ then $a=b=c$ so $2\lambda_a^a = 0 \Rightarrow \lambda_a^a = 0$

If $a \neq b$ then $\lambda_a^b + \lambda_b^a = 0$]

✓

From i) since $\{U_a\}$ is a local basis of vector fields with dual basis of one forms $\{\theta^a\}$,

$L_X\theta^1 = \mu_a^1\theta^a = \mu_1^1\theta^1 + \mu_2^1\theta^2$

$= -\lambda_1^1\theta^1 + \lambda_1^2\theta^2$, also from i)

$L_X\theta^2 = \mu_1^2\theta^1 + \mu_2^2\theta^2$

$= -\lambda_2^1\theta^1 - \lambda_2^2\theta^2$

(λ_b^a) skew symmetric $\Rightarrow \lambda_1^1 = \lambda_2^2 = 0$

$\Rightarrow L_X\theta^1 = -\lambda_1^2\theta^2, \quad L_X\theta^2 = -\lambda_2^1\theta^1$

$= \lambda_2^1\theta^2$

Put $\lambda_2^1 = \lambda$ ✓

Then $L_X\theta^1 = \lambda\theta^2, \quad L_X\theta^2 = -\lambda\theta^1$ / 5

iv) The 1st structure equations are

$d\theta^1 + \omega_a^1 \wedge \theta^a = 0$

$d\theta^2 + \omega_a^2 \wedge \theta^a = 0$

Using the first

$L_X(d\theta^1) + L_X(\omega_a^1)\wedge\theta^a + \omega_a^1 \wedge L_X(\theta^a) = 0$