M 827 TMA 03 2001 Solutions

Question 1

(i) $(\mathbf{v}-\mathbf{w}).(\mathbf{v}+\mathbf{w}) = \mathbf{v}.\mathbf{v}-\mathbf{w}.\mathbf{w} = 0 \Leftrightarrow |\mathbf{v}| = |\mathbf{w}|$.

(ii) $g(f_1, e_3) = 0 = k_1 g(e_1, e_3) + g(e_3, e_3) = 2k_1 + 3 \Rightarrow k_1 = -3/2$.

Similarly $k_2 = -3/2$.

So $f_1 = -(3/2)e_1 + e_3$ and $f_2 = -(3/2)e_2 + e_3$ are both orthogonal to e_3.

Check that they form a basis for the 2-dimensional orthogonal complement to e_3. $g(f_1, f_1)$

$= \frac{9}{4} g(e_1, e_1) - 3\, g(e_1, e_3) + g(e_3, e_3) = 3/2$, so $|f_1| = \sqrt{3/2}$.

Similarly $|f_2| = \sqrt{3/2}$.

Similarly $g(f_1, f_2) = -3/4$, so $\cos\theta = -1/2$ (and $\theta = 2\pi/3$).

(iii) $f_1 - f_2$ and $f_1 + f_2$ are orthogonal (by (i)), so an orthonormal basis is found by normalising $f_1 - f_2$, $f_1 + f_2$ and e_3:

From $f_1 - f_2$ we obtain $\hat{e}_1 = \frac{1}{\sqrt{2}}(e_2 - e_1)$. From $f_1 + f_2$ we obtain

$\hat{e}_2 = \frac{-3}{\sqrt{2}}(e_1 + e_2) + 2\sqrt{\frac{2}{3}}e_3$. From e_3 we obtain $\hat{e}_3 = \frac{1}{\sqrt{3}}e_3$.

(iv) The volume form Ω is a positive multiple, k say, of $\theta_1 \wedge \theta_2 \wedge \theta_3$.

$k(\theta_1 \wedge \theta_2 \wedge \theta_3)(\hat{e}_1, \hat{e}_2, \hat{e}_3) = \pm 1$. So (after a calculation) $k = 1$, so the $\hat{e}$ basis is positively oriented with respect to the e basis.

Or argue that the volume form is $\sqrt{\det(g_{ab})}\theta_1 \wedge \theta_2 \wedge \theta_3$.

(v) Since the $\hat{e}$ basis is positively oriented, $*\hat{e}^3 = \hat{e}^1 \wedge \hat{e}^2$.

$\hat{e}^1 = \frac{1}{\sqrt{2}}(e^2 - e^1)$, etc, because raising indices by g is linear map, and similarly,

$\hat{e}^2 = \frac{-3}{\sqrt{2}}(e^1 + e^2) + 2\sqrt{\frac{2}{3}}e^3$, and $\hat{e}^3 = \frac{1}{\sqrt{3}}e^3$.

So $*e^3 = *(\sqrt{3}\hat{e}^3) = \sqrt{3}*(\hat{e}^3) = \sqrt{3}(\hat{e}^1 \wedge \hat{e}^2) =$

$3e^1 \wedge e^2 - 2e^1 \wedge e^3 + 2e^2 \wedge e^3$.