(ii) It's enough to state the obvious properties of $L_U g$ needed for a tensor field of type (0,2), and show that it is $F(S)$ multilinear. For that, by symmetry, looking at one factor is enough. For that, $L_U g\,(fV,W)$ = three terms. Show that you get three terms for $fL_U g\,(V,W)$ and two terms involving $(Uf)g(V,W)$ which cancel.

(iii) For a Killing field $L_X g(U_a,U_b)=0$, $a,b=1,2$, so

$$X(g(U_a,U_b)) = g([X,U_a],U_b)+g(U_a,[X,U_b]).$$

But $g(U_a,U_b)=0$ or 1, so X is a Killing field if and only if RHS = 0, i.e. (on working it out) if and only if

$$\lambda_a^c\delta_{cb}+\delta_{ac}\lambda_b^c=0.$$

From (i) $L_X\theta^a=-\lambda_b^a\theta^b$, and by the skew-symmetry each $\lambda_a^a=0$ (no summation) and $\lambda_1^2=-\lambda_2^1=\lambda$, say, whence the desired result.

(iv) The structure equations are $d\theta^1=\omega\wedge\theta^2$, $d\theta^2=-\omega\wedge\theta^1$.

Compute $d(L_X\theta^1)=L_X d\theta^1$ (d and L_X commute).

The RHS = $L_X(\omega\wedge\theta^2)=L_X\omega\wedge\theta^2+\omega\wedge L_X\theta^2=L_X\omega\wedge\theta^2-\lambda\omega\wedge\theta^1$

The LHS = $d(\lambda\theta^2)=d\lambda\wedge\theta^2+\lambda d\theta^2=d\lambda\wedge\theta^2-\lambda\omega\wedge\theta^1$.

So $L_X\omega\wedge\theta^2=d\lambda\wedge\theta^2$. Similarly, $L_X\omega\wedge\theta^1=d\lambda\wedge\theta^1$, whence (because θ^1 and θ^2 are a basis) $L_X\omega=d\lambda$.

(iv) Since $d\omega=-K\theta^1\wedge\theta^2$, we have, (by (iv)) $L_X d\omega=0=L_X(-K\theta^1\wedge\theta^2)$. Show this last expression = $-X(K\theta^1\wedge\theta^2)$, whence $X(K)=0$.

Question 4

(Enneper's surface)

Writing u for ξ^1 and v for ξ^2, the parametric form of the surface is

$$(u,v)\mapsto\left(u+uv^2-u^3/3,-v-u^2v+v^3/3,u^2-v^2\right)=\mathbf{r}(u,v)$$

We compute

$$\mathbf{r}_u=\left(1+v^2-u^2,-2uv,2u\right)$$

$$\mathbf{r}_v=\left(2uv,-1-u^2+v^2,-2v\right)$$

the normal vector $r_u\times r_v=\left(2ud,2vd,(u^2+v^2-1)d\right)$, where $d=1+u^2+v^2$

the unit normal vector