

3) $\langle U_a, \Theta^b \rangle = \delta_a^b$ (15)

$$V \langle U_a, \Theta^b \rangle = \langle L_V U_a, \Theta^b \rangle + \langle U_a, L_V \Theta^b \rangle = 0$$

$$\lambda_a^c \langle U_c, \Theta^b \rangle + \mu_a^b \langle U_a, \Theta^d \rangle = 0$$

$$\lambda_a^c \delta_c^b + \mu_a^b \delta_a^d = 0$$

When $c=b, d=a$

$$\lambda_a^b + \mu_a^b = 0$$

$$\lambda_a^b = -\mu_a^b = -(\mu_a^b)^T \Rightarrow \mu_a^b = -(\lambda_a^b)^T$$

$$\Rightarrow L_V U_a = \lambda_a^b U_b$$

$$L_V \Theta^a = \mu_a^b \Theta^b = -(\lambda_a^b)^T \Theta^b$$

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ii) $L_{Uf} g(V, W) = Uf g(V, W) - g([U, fV], W)$

$$= Uf g(V, W) - g(Uf V + f[U, V], W)$$

$$= Uf g(V, W) - g(Uf V, W) - f g([U, V], W)$$

$$= Uf g(V, W) + f U(g(V, W) - (Uf)g(V, W))$$

$$- f g([U, V], W) - f g(V, [U, W])$$

$$= f(Ug(V, W) - g([U, V], W) - g(V, [U, W]))$$

$$= f L_{Uf} g(V, W)$$

$$L_{Uf} g(V, W) = U(g(V, fW)) - g([U, V], fW)$$

$$- g(V, [U, fW])$$

$$= U(f g(V, W)) - f g([U, V], W)$$

$$- g(V, (Uf)W + f[U, W])$$

$$= (Uf) g(V, W) + f U(g(V, W)) - f g([U, V], W)$$

$$- (Uf) g(V, W) - f g(V, [U, W])$$

$$= f(Ug(V, W) - g([U, V], W) - g(V, [U, W]))$$

as required.

$L_{Uf} g$ has two vector args \Rightarrow is a type $(0,2)$ -tensor.

iii) For X to be a Killing field,

$$L_X g = 0.$$