Please make sure that the assignment number is correctly entered on your PT3 form as

$\boxed{\text{M827 03}}$.

Question 1 (25 marks)

(i) Show that, in any Euclidean vector space, if vectors v and w have the same length then $v + w$ and $v - w$ are orthogonal. [2]

$\mathcal{V}$ is a 3-dimensional vector space, $\{e_a\}$, $a = 1, 2, 3$ is a basis for $\mathcal{V}$ and $\{\theta^a\}$ the dual basis for $\mathcal{V}^*$. A scalar product g on $\mathcal{V}$ has

$$\begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \end{pmatrix}$$

for the matrix of its components g_{ab} with respect to the basis $\{e_a\}$.

(ii) Show that constants k_1 and k_2 can be found such that, if

$$f_1 = k_1 e_1 + e_3 \quad \text{and} \quad f_2 = k_2 e_2 + e_3$$

then $\{f_1, f_2\}$ is a basis for the orthogonal complement to the subspace of $\mathcal{V}$ (of dimension 1) spanned by the vector e_3. Calculate the lengths of the vectors f_1 and f_2, and the cosine of the angle between them. [6]

(iii) Construct a basis $\{\hat{e}_a\}$ for $\mathcal{V}$, orthonormal with respect to the scalar product g, such that $\hat{e}_3$ is a scalar multiple of e_3. [6]

(iv) Find the volume form on $\mathcal{V}$ determined by the scalar product g for which the given basis $\{e_1, e_2, e_3\}$, in the given order, is positively oriented. Express your answer in terms of the dual basis $\{\theta^a\}$. [3]

(v) Let $e^a = g(e_a)$, where $g\colon \mathcal{V} \to \mathcal{V}^*$ is the linear map determined by the scalar product. Find $*e^3$, expressing your answer in terms of the e^a and their exterior products. [8]