Question 3 (20 marks)

This question is concerned with the extension of the definition of a Killing field to surfaces, and with the properties of Killing fields on surfaces.

(i) Let $\{U_a\}$, $a = 1, 2$, be a local basis of vector fields on a surface $\mathcal{S}$, with dual basis of 1-forms $\{\theta^a\}$. Note that, since $\{U_a\}$ is a local basis, for any vector field V on $\mathcal{S}$ there are functions λ^a_b such that $[V, U_a] = \lambda^b_a U_b$; and similarly there are functions μ^b_a such that $\mathcal{L}_V \theta^a = \mu^a_b \theta^b$. Find the relation between λ^b_a and μ^a_b. [3]

The Lie derivative $\mathcal{L}_U g$ of the metric g on a surface $\mathcal{S}$ with respect to a vector field U on $\mathcal{S}$ is defined by setting

$$\mathcal{L}_U g(V, W) = U(g(V, W)) - g([U, V], W) - g(V, [U, W]),$$

where V and W are arbitrary (local) vector fields, in analogy with the case of an affine space.

(ii) Show that, for any vector field U on $\mathcal{S}$, $\mathcal{L}_U g$ has the $\mathcal{F}(\mathcal{S})$-multilinearity property of a type $(0, 2)$ tensor field. [4]

A vector field X on the surface for which $\mathcal{L}_X g = 0$ is called a Killing field.

(iii) Suppose there is given a local orthonormal basis of vector fields on $\mathcal{S}$, say $\{U_a\}$, $a = 1, 2$, with dual basis of 1-forms $\{\theta^a\}$. For any vector field X on $\mathcal{S}$, set $[X, U_a] = \lambda^b_a U_b$. Show that the necessary and sufficient condition for X to be a Killing field is that

$$\delta_{ac}\lambda^c_b + \delta_{bc}\lambda^c_a = 0,$$

so that the matrix (λ^a_b) is skew-symmetric. Deduce that when X is a Killing field there is a function λ such that

$$\mathcal{L}_X \theta^1 = \lambda\theta^2 \quad \text{and} \quad \mathcal{L}_X \theta^2 = -\lambda\theta^1.$$ [5]

(iv) By taking the Lie derivative of both sides of the first structure equations for $\mathcal{S}$, show that if X is a Killing field then

$$\mathcal{L}_X \omega = d\lambda,$$

where ω is the connection 1-form for the given basis. [5]

(v) Deduce that the Gaussian curvature is constant along the integral curves of any Killing field on a surface. [3]

You may assume that the Lie derivative of forms on a surface has the same properties as the corresponding operator in affine space.

Question 4 (15 marks)

A surface (Enneper's surface) is given parametrically by

$$(\xi^1, \xi^2) \mapsto (\xi^1 + \xi^1 (\xi^2)^2 - (\xi^1)^3/3,\ -\xi^2 - (\xi^1)^2 \xi^2 + (\xi^2)^3/3,\ (\xi^1)^2 - (\xi^2)^2),$$

where $(\xi^1)^2 + (\xi^2)^2 < \infty$. Show that the mean curvature of this surface is zero everywhere.

$\delta \quad \lambda^a_b +$ $\qquad \delta_{ab}\lambda^b_b + \delta_{bc}\lambda^c_a = 0$

$\lambda^a_a + \lambda^a_a = 0$