Please make sure that the assignment number is correctly entered on your PT3 form as

$\boxed{\text{M827 02}}$.

Question 1 – 15 marks

(i) Let X and Y be vector fields and ω_1 and ω_2 be 1-forms. Use the fact that

$$L_V(\omega_1 \wedge \omega_2) = L_V\omega_1 \wedge \omega_2 + \omega_1 \wedge L_V\omega_2$$

to prove the fundamental formula for the Lie derivative L_V:

$$(L_V(\omega_1 \wedge \omega_2))(X,Y) =$$
$$L_V((\omega_1 \wedge \omega_2)(X,Y)) - (\omega_1 \wedge \omega_2)(L_V X, Y) - (\omega_1 \wedge \omega_2)(X, L_V Y).$$ [4]

The purpose of the rest of this question is to make you familiar with expressions for the interior product and Lie derivative in low-dimensional cases. Throughout this question, ∂_1, ∂_2 are coordinate vector fields on a 2-dimensional affine space and ω_1 and ω_2 are a dual basis of 1-forms. Also, a_1, b_1, b_2, c_1, c_2 and f are functions of the coordinates, $V = a_1\partial_1$ and $X = c_1\partial_1 + c_2\partial_2$ are vector fields.

(ii) The fundamental formula for the interior product $\rfloor$ in this context is

$$W\rfloor(\chi_1 \wedge \chi_2) = (W\rfloor\chi_1) \wedge \chi_2 - \chi_1 \wedge (W\rfloor\chi_2),$$

where W is a vector field and χ_1 and χ_2 are 1-forms. Verify this formula in the special case

$$V\rfloor(\chi_1 \wedge \chi_2) = (V\rfloor\chi_1) \wedge \chi_2 - \chi_1 \wedge (V\rfloor\chi_2),$$

where $\chi_1 = b_1\omega_1$ and $\chi_2 = b_2\omega_2$, by evaluating each side separately on a vector field X. (There is no need to express X with respect to a basis.) [4]

(iii) Verify that

$$L_V(f\omega_1) = (Vf)\omega_1 + fL_V(\omega_1)$$

by evaluating each side separately on a vector field X. (There is no need to express X with respect to a basis.) [3]

(iv) The fundamental formula connecting the Lie derivative and the exterior derivative is $L_V\omega = d(V\rfloor\omega) + V\rfloor d\omega$ (for arbitrary vector fields V and 1-forms ω). Verify it in the special case, where $V = a_1\partial_1$ and $\omega = b_1\omega_1$ by evaluating both sides on the vector field $X = c_1\partial_1 + c_2\partial_2$. [4]