**Question 2** (15 marks)

Let $\{e_a\}$, $a = 1, 2, 3, 4$, be a basis for a 4-dimensional vector space $\mathcal{V}$, and let $\{\theta^a\}$ be the dual basis for $\mathcal{V}^*$. Let

$$\alpha = 3\theta^1 - 2\theta^2 + \theta^3 - \theta^4,$$
$$\beta = \theta^1 \wedge \theta^2 + \theta^1 \wedge \theta^3 - \theta^2 \wedge \theta^4,$$
$$v_1 = e_1 + 2e_2 - 3e_3,$$
$$v_2 = e_1 + e_3 - e_4,$$
$$v_3 = e_2 + e_4,$$
$$v_4 = e_3 - 2e_4.$$

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Compute the following.

(i) $\alpha \wedge \beta$ [3]

(ii) $v_1 \lrcorner \beta$ [4]

(iii) $\beta(v_2, v_3)$ [4]

(iv) $(\beta \wedge \beta)(v_1, v_2, v_3, v_4)$ [4]

**Question 3** (20 marks)

A *vector-valued 3-form* on a vector space $\mathcal{V}$ is a function which (like a 3-form) takes 3 vector arguments and is multilinear and alternating in them; but (unlike a 3-form) takes its values in $\mathcal{V}$. In other words, a vector-valued 3-form is an alternating, multilinear map $L\colon \mathcal{V}^3 \to \mathcal{V}$.

(i) Show that the set of vector-valued 3-forms on a vector space $\mathcal{V}$ is itself a vector space. [6]

(ii) Let $w$ be a fixed element of $\mathcal{V}$ and $\alpha$ an ordinary 3-form. Define $w \otimes \alpha$ as follows: for any $v_1, v_2, v_3 \in \mathcal{V}$, set

$$(w \otimes \alpha)(v_1, v_2, v_3) = \alpha(v_1, v_2, v_3)w.$$

Show that $w \otimes \alpha$ is a vector-valued 3-form. [3]

(iii) Let $L$ be a vector-valued 3-form and $\theta$ an element of $\mathcal{V}^*$. Define $\theta_L$ as follows: for any $v_1, v_2, v_3 \in \mathcal{V}$, set

$$\theta_L(v_1, v_2, v_3) = \langle L(v_1, v_2, v_3), \theta \rangle.$$

Show that $\theta_L$ is an ordinary 3-form. [3]

(iv) Let $\{e_a\}$ be a basis for $\mathcal{V}$ and $\{\theta^a\}$ the dual basis for $\mathcal{V}^*$. Show that, for any vector-valued 3-form $L$,

$$\sum_a e_a \otimes \theta^a_L = L.$$ [6]

(v) Find the dimension of the space of vector-valued 3-forms on $\mathcal{V}$, if the dimension of the vector space $\mathcal{V}$ is 4. [2]

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$$L\begin{pmatrix} v_1 & u_1 & w_1 \\ v_2 & u_2 & w_2 \\ v_3 & u_3 & w_3 \\ v_4 & u_4 & w_4 \end{pmatrix} = \sum \theta \wedge \theta \wedge \theta_k$$

3×4  4×3  3×  3×1

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