An $\mathbf{R}$-linear operator D on forms (i.e. an a ... dimension m) is said to be of *degree* k if it maps p-forms to $(p+k)$- ... s, $p = 0, 1, 2, \ldots, m$; it is a *derivation* if for every pair of forms α, β,

$$D(\alpha \wedge \beta) = (D\alpha) \wedge \beta + \alpha \wedge (D\beta),$$

and an *anti-derivation* if

$$D(\alpha \wedge \beta) = (D\alpha) \wedge \beta + (-1)^p \alpha \wedge (D\beta)$$

(where α is a p-form). (Both instances are supposed to include the case where α, say, is a 0-form, i.e. a function; the wedge is then really superfluous, since it represents ordinary multiplication of a form by a function.) It may be assumed that $D\alpha = 0$ if $p + k < 0$ or $p + k > m$ (where α is a p-form and k is the degree of D).

(i) In each of the following cases, show (by quoting the relevant properties from the text) that the operator defined is a derivation or an anti-derivation, and state its degree.

(a) $D\alpha = d\alpha$. 1

(b) $D\alpha = \mathcal{L}_V\alpha$, where V is a fixed vector field. zero

(c) $D\alpha = V \lrcorner \alpha$, where V is a fixed vector field. −1 [6]

(ii) Show that a derivation is uniquely determined by its action on 0-forms (functions) and 1-forms. That is, show that if D_1 and D_2 are two operators of the same degree, which are both derivations, and

$$D_1(f) = D_2(f) \quad \text{for all } f \in \mathcal{F}(\mathcal{A}),$$
$$D_1(\theta) = D_2(\theta) \quad \text{for all 1-forms } \theta \text{ on } \mathcal{A},$$

then

$$D_1(\alpha) = D_2(\alpha)$$

for all forms α. [8]

(iii) Let D_1 and D_2 be anti-derivations of degrees k_1 and k_2 respectively, where k_1 and k_2 are both odd. Show that the operator D defined by

$$D\alpha = D_1(D_2\alpha) + D_2(D_1\alpha)$$

is a derivation of degree $k_1 + k_2$. [3]

(iv) Show that if V is any vector field on $\mathcal{A}$, and D_V is defined by

$$D_V\alpha = V \lrcorner (d\alpha) + d(V \lrcorner \alpha),$$

then D_V is a derivation of degree 0. Deduce *from the previous parts of this question* that D_V is the Lie derivative with respect to V. [8]