Question 4 (20 marks)

Let $\mathcal{A}$ be a 3-dimensional affine space with local coordinates (x^1, x^2, x^3) and let $\mathcal{B}$ be a 2-dimensional affine space with local coordinates (y^1, y^2). Let ω be the 2-form on $\mathcal{A}$ given by

$$\omega = (x^1 + x^2)dx^1 \wedge dx^2 - (x^1 + x^3)dx^1 \wedge dx^3 + (x^2 + x^3)dx^2 \wedge dx^3.$$

(i) A smooth map $\phi : \mathcal{B} \to \mathcal{A}$ is given in terms of the local coordinates by

$$(y^1, y^2) \mapsto (y^1 y^2, (y^1)^2 - (y^2)^2, (y^1)^2 + (y^2)^2).$$

Compute $\phi^* \omega$. [5]

(ii) Compute $\mathcal{L}_V \omega$, where V is the vector field given by

$$V = x^3 \frac{\partial}{\partial x^2} + x^2 \frac{\partial}{\partial x^3}.$$ [5]

(iii) Show that $d\omega = 0$, and find a 1-form θ on the coordinate patch in $\mathcal{A}$ such that $\omega = d\theta$. [5]

(iv) Find a vector field W such that

$$\omega = W \lrcorner \Omega, \qquad \text{where} \quad \Omega = dx^1 \wedge dx^2 \wedge dx^3.$$

What can you conclude about W from the fact that $d\omega = 0$? [5]

$V_y = W_y^a e_a$

$\omega(y) = \Sigma_{ey} W_y^a e_a(y)$

$AA^{-1} = 1$

$(dA)A^{-1} + A\,d(A^{-1}) = 0$

$\tan\theta = y/x$

$\sec^2\theta \frac{d\theta}{dx} = -\frac{y}{x^2}$

$\frac{\partial\theta}{\partial x} = -\frac{\sin\theta}{r}$

$\frac{\partial\theta}{\partial y} = \frac{\cos\theta}{r}$

$u_1 = -\sin\theta\cos\theta \frac{\partial}{\partial\theta}$

$u_2 = \frac{\partial}{\partial\theta}$