(i) The hard part in this question is to see what is to be proved, which is that sums of the form $\alpha_1 L_1 + \alpha_2 L_2$, where the α's are constants and the L's are alternating multilinear vector-valued maps to V, are also are alternating multilinear vector-valued maps to V. Once this is done the fact that the set of them plainly forms some sort of subset of a vector space makes the rest automatic. (To see that there is something to prove, consider the set of all functions from **R** to **R** which never take the value zero . . .).

(ii) Routine using the definition of the map and the definitions of alternating and multilinear.

(iii) Routine using the definition of the map and the definitions of alternating and multilinear. It's enough to check multilinearity in one factor, provided you say so.

(iv) Writing out what you know may be difficult. I did it by observing that it is enough to evaluate each side on a triple $\left(e_i, e_j, e_k\right)$, and writing $L\left(e_i, e_j, e_k\right) = \sum_{a} \lambda^{a}_{ijk} e_a$. The calculation is again routine.

(v) There are two ways to do this. Either observe that an alternating map on V^3 is of dimension $(1/6)n(n-1)(n-2)$, if V is of dimension n, and use the fact that the dimension of the space of linear maps from one vector space to another is the product of the dimensions, so the required dimension is dimension $(1/6)n(n-1)(n-2)n = (1/6)n^2(n-1)(n-2)$. Or use (iv), noting that there are n choices for the e and $(1/6)n(n-1)(n-2)$ choices for the θ. Again, the required dimension is $(1/6)n^2(n-1)(n-2)$. When $n = 4$, this number is 16.

Question 4

(i) This is a straight-forward calculation, once you realise it says nothing you didn't learn at school. Use $x^1 = y^1 y^2$ etc., to get $dx^1 = dy^1 y^2 + y^1 dy^2$, etc. Put it all together and deduce $\phi^* \omega = 4y^1 y^2\left(4y^{1^2} - y^{2^2}\right)dy^1 \wedge dy^2$.

(ii) This follows from a formula in C&P, p. 127. You need to write the components of V as V^i, $i = 1, 2, 3$, so $V^1 = 0$, $V^2 = x^3$, and $V^3 = x^2$. The answer is

$$L_V \omega = -x^1 dx^1 \wedge dx^2 + \left(x^2 + x^3\right) dx^2 \wedge dx^3 + x^1 dx^1 \wedge dx^3 .$$

(iii) It's routine that $d\omega = 0$. To find a suitable θ you can either spot one or use the method in C&P. Either way, and certainly if you going spotting, check the answer. Note that the answer is not unique. One answer is $\theta = \left(x^1x^3 - x^1x^2\right)dx^1 + \left(x^1x^2 - x^2x^3\right)dx^2 + \left(x^2x^3 - x^1x^3\right)dx^3$. All others differ from this one by a form $\psi = dg$ for some function g .

(iv) Set $W = a_1\partial_1 + a_2\partial_2 + a_3\partial_3$ and compute $W \rfloor \Omega$. You soon find $a_i = x^j + x^k \,(i \neq j \neq k \neq i)$. So $L_W\Omega = d\left(W \rfloor \Omega\right) = 0$, and the conclusion is that $L_W\Omega = 0$ and the flow associated to W is volume-preserving.

Question 5