TMA M824 03 **Cut-off date** 12 July 2000

Please make sure that the assignment number is correctly entered on your PT3 form as

$$\boxed{\text{M824 03}}.$$

This TMA is based on Block III of the course.

In this TMA all the metric spaces considered are subspaces of $\mathbb{R}^N$ (for various values of N). Thus 'A is compact' is the same as 'A is closed and bounded'.

If a typical element x of the space A is given by

$$x = (x_1, x_2, \ldots, x_m)$$

where $A \subset \mathbb{R}^m$, $\mathbb{R}^m = \{(x_1, x_2, \ldots, x_m) \mid x_i \in \mathbb{R},\ i = 1, \ldots, m\}$, and a typical element y of the space B is given by

$$y = (y_1, y_2, \ldots, y_n)$$

where $B \subset \mathbb{R}^n$, $\mathbb{R}^n = \{(y_1, y_2, \ldots, y_n) \mid y_i \in \mathbb{R},\ i = 1, \ldots, n\}$, then the product space $A \times B \subset \mathbb{R}^{m+n}$ has typical element

$$(x, y) \equiv (x_1, \ldots, x_m, y_1, \ldots, y_n).$$

Question 1 – 10 marks

The space G is *path-connected* if, for any $x, y \in G$, there exists a continuous map $\gamma : [0,1] \longrightarrow G$ with $\gamma(0) = x$ and $\gamma(1) = y$ (called a *path* in G). (This property is simply referred to as *connected* by Curtis on page 79.)

(i) Show that $GL(n, \mathbb{C})$ is path-connected.

[*Hint*: Use the Jordan normal form, *Course Notes*, page 11, Note 7.]

(ii) Is $GL(n, \mathbb{R})$ path-connected? Justify your answer.

Question 2 – 10 marks

Let U, V be subsets of $\mathbb{R}^m, \mathbb{R}^n$ respectively.

(i) If U and V are open, show that $U \times V \subset \mathbb{R}^{n+m}$ is open.

(ii) If U and V are compact, show that $U \times V$ is compact.

(iii) Show that the space $U(2)$ is *not* homeomorphic to the matrix group

$$U(1,1) \equiv \left\{ u \in M_2(\mathbb{C}) \;\middle|\; u \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} u^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\},$$

where $u^\dagger = {}^t\overline{u}$ is the Hermitian conjugate of u.

Question 3 – 10 marks

(i) Show that

$$T = \left\{ \begin{bmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{bmatrix} \;\middle|\; \theta_1, \theta_2 \in \mathbb{R} \right\}$$

is a maximal torus in the matrix group $U(1,1)$ (defined in Question 2(iii) above).

(ii) Hence or otherwise find the centre of $U(1,1)$.

(iii) Find a maximal torus in $SU(1,1)$ and the centre of $SU(1,1)$, where

$$SU(1,1) \equiv \{u \in U(1,1) \mid \det u = 1\}.$$