TMA 02 M824. ①

1) i) $B'(t) = \frac{d}{dt}\left(e^{tA} B e^{-tA}\right)$

$= Ae^{tA}Be^{-tA} + e^{tA}B(-Ae^{-tA})$

$= AB(t) - e^{tA}Be^{-tA}A$

$= AB(t) - B(t)A = [A, B(t)]$ PTO.

Where ~~I~~ have used the product rule for differentiating a product.

ii) $B(t) = \sum_{m=0}^{\infty} \frac{t^m}{m!} B_m$. Coefficient of $t^0 = B_0 = \frac{B}{0!}$, t is $[A,B] = \frac{B_1}{1}$

Prove by induction. Assume Coefficient of t^m is $\frac{B_m}{m!}$

Expand $B(t) = e^{tA}Be^{-tA}$ in terms of Taylor series

$$B(t) = \left(I + tA + \frac{t^2A^2}{2!} + \frac{t^3A^3}{3!} \ldots\right) B \left(I - tA + \frac{t^2A^2}{2!} - \frac{t^3A^3}{3!} \ldots\right)$$

$$= B + t(AB - BA) + t^2\left(\frac{A^2B}{2} - ABA + \frac{BA^2}{2}\right) + \ldots$$

$$+ \sum_{n=0}^{m} \frac{t^{m-n}A^{m-n}}{(m-n)!} \frac{B t^n A^n (-1)^n}{n!} \ldots$$

(since the general term is a Cauchy Sum over powers of tA).

Concentrate on the general term

$$\sum_{n=0}^{m} \frac{t^{m-n}}{(m-n)!} \frac{A^{m-n} B t^n A^n (-1)^n}{n!} = \frac{t^m}{m!} \sum_{n=0}^{m} \frac{m!}{(m-n)!\,n!} A^{m-n} B A^n (-1)^n$$

Now we have to prove

$$B_m = \sum_{n=0}^{m} \frac{m!}{(m-n)!\,n!} A^{m-n} B A^n (-1)^n$$