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Dear M824 Student:

Notes on TMA 02

Continuing my series of Notes on the assignments, which I hope will bring out some general points I have noted to the attention of all, as well as illustrating the general context of Physics, where some of the applications are to be found.

Question 1: The Campbell-Baker-Hausdorff formula tells you how to express the product of the exponentials of two matrices as an exponential of a third matrix. Thus,

$$\exp(A)\exp(B)=\exp(A+B+(1/2)[A,B]+\ldots..)$$

where the remarkable fact is that all the subsequent terms are expressible as commutators. This means that the sum is sitting in a Lie algebra (and that the exponential is consequently in a Lie group). Our result is equivalent to neglecting all but the terms explicitly written above. Such formulae are very important in Quantum Mechanics, which often has to treat exponents of operators (matrices). In fact, matrices are used extensively in Quantum Mechanics, the most successful description that we have of the microscopic world; indeed, Quantum Mechanics was once called Matrix Mechanics. Matrix groups (Lie groups) also play an important role in the description of the symmetries of nature.

The only mathematical point to note is that a matrix A commutes with any function of itself, f(A).

Question 2: One of the two great theories of Nature is Relativity (Quantum Mechanics is the other), and the Special Theory of Relativity describes the motion of fast-moving objects. This motion mixes time (x_0) and space (x_1,x_2,x_3) in such a way as to keep the form $x_0^2-x_1^2-x_2^2-x_3^2$ invariant (constant). The Group of Transformations (Matrices) which preserve this invariance is called the Lorentz Group. Our question describes an alternative way of introducing the Lorentz group, a method much favoured by physicists. The last, and hardest part, of the question, viz. to show that u did not give a reflection, follows immediately by noting that (using multiplication)

$$u\,\tau_2 u^\dagger = \tau_2 \text{ and } u\,\tau_3 u^\dagger = \tau_3 \Rightarrow u\,\tau_1 u^\dagger = \tau_1.$$

Later on we shall see (topology) that SU(2) is a connected set, while O(3) consists of two connected pieces; since (iv) essentially gives us a smooth map from SU(2) to O(3), we can only get to the connected - i.e. SO(3) - bit (C81, Prop.4).

Question 3: (ii) Finding representations of Lie Algebras is one of the most important aspects of the application of these systems in Physics. For example, it turns out that the Elementary Particles (the 'building blocks of nature') may be thought of as representations of the Lorentz Group - an idea due to the late Eugene Wigner. The sets of matrices that we deal with in this course may be thought of as concrete realisations of the abstract Lie groups; thus the elements (unitary 2X2 matrices of determinant 1) and multiplication of SU(2) can be thought of as defining the *abstract* Lie group SU(2); with underlying Lie algebra su(2). Another representation of su(2) would be given by ad - but in this case it would be in terms of 3X3 matrices as the vector space of su(2) is 3-dimensional.
(iii)The abstract definition of a Lie Algebra is that of a linear space with a 'bracket' operation satisfying the usual rules (antisymmetry and Jacobi identity). One does not need to define [X,Y] as XY-YX, and indeed the ordinary multiplication XY need not be defined at all. However, in *our* Course everything is