(12)

4) i) Let $\underline{q} = a + b\underline{i} + c\underline{j} + d\underline{k}$. Then $\rho(q)$ is a linear transformation, since if $r \in \mathbb{R}$

$\rho(q)(r\underline{x}) = q r\underline{x}\bar{q} = r q\underline{x}\bar{q} = r\rho(q)(\underline{x})$

since r commutes with all quaternions. Also

$\rho(q)(\underline{x}+\underline{y}) = q(\underline{x}+\underline{y})\bar{q}$

$= q\underline{x}\bar{q} + q\underline{y}\bar{q}$

$= \rho(q)(\underline{x}) + \rho(q)(\underline{y})$.

$\therefore$ We need only consider the effect of $\rho(q)$ on the basis elements of S^3

$\rho(q)(a) = qa\bar{q} = aq\bar{q} = a$

$\rho(q)(\underline{i}) = (a+b\underline{i}+c\underline{j}+d\underline{k})\underline{i}(a-b\underline{i}-c\underline{j}-d\underline{k})$

$= (a+b\underline{i}+c\underline{j}+d\underline{k})(a\underline{i}+b-c\underline{k}+d\underline{j})$

$= (ab-ab-cd+cd) + \underline{i}(a^2+b^2-c^2-d^2)$

$+ \underline{j}(bc+ad+bc+ad) + \underline{k}(-ac+bd-ac+bd)$

$= \underline{i}(a^2+b^2-c^2-d^2) + \underline{j}(2bc+2ad) + \underline{k}(2bd-2ac)$

$\rho(q)(\underline{j}) = (a+b\underline{i}+c\underline{j}+d\underline{k})\underline{j}(a-b\underline{i}-c\underline{j}-d\underline{k})$

$= (a+b\underline{i}+c\underline{j}+d\underline{k})(a\underline{j}+b\underline{k}+c-d\underline{i})$

$= \underline{i}(-ad+bc+bc-ad) + \underline{j}(a^2-b^2+c^2-d^2)$

$+ \underline{k}(ab+ab+cd+cd)$

$= \underline{i}(2bc-2ad) + \underline{j}(a^2-b^2+c^2-d^2) + \underline{k}(2ab+2cd)$

$\rho(q)(\underline{k}) = (a+b\underline{i}+c\underline{j}+d\underline{k})\underline{k}(a-b\underline{i}-c\underline{j}-d\underline{k})$

$= (a+b\underline{i}+c\underline{j}+d\underline{k})(a\underline{k}-b\underline{j}+c\underline{i}+d)$

$= \underline{i}(ac+bd+ac+bd) + \underline{j}(-ab-ab+cd+cd)$

$+ \underline{k}(a^2-b^2-c^2+d^2)$

$= \underline{i}(2ac+2bd) + \underline{j}(-2ab+2cd) + \underline{k}(a^2-b^2-c^2+d^2)$

$\therefore$ If $\alpha = x_0 + x_1\underline{i} + x_2\underline{j} + x_3\underline{k}$

$\rho(q)(\alpha) = x_0 + \underline{i}(x_1(a^2+b^2-c^2-d^2) + 2x_2(bc-ad) + 2x_3(ac+bd))$

$+ \underline{j}(2x_1(bc+ad) + x_2(a^2-b^2+c^2-d^2) + 2x_3(-ab+cd))$

$+ \underline{k}(2x_1(bd-ac) + 2x_2(ab+cd) + x_3(a^2-b^2-c^2+d^2))$

not clear