(2)

Prove by induction. Suppose true $n=k$

$$B_k = \sum_{n=0}^{k} \frac{(-1)^n k!}{(k-n)!\,n!} A^{k-n} B A^n$$

Try $n=k+1$

$$B_{k+1} = [A, B_k] = \left[A, \sum_{n=0}^{k} \frac{(-1)^{\cancel{k+n}\,n} k!}{(k-n)!\,n!} A^{k-n} B A^n\right]$$

$$= \sum_{n=0}^{k} \left(\frac{k!(-1)^{\cancel{k+n}}}{(k-n)!\,n!} A^{k+1-n} B A^n - \frac{k!(-1)^{\cancel{k+n}}}{(k-n)!\,n!} A^{k-n} B A^{n+1} \right)$$

$$= \sum_{n=0}^{k} \frac{k!(-1)^n}{(k-n)!\,n!} A^{k+1-n} B A^n + \frac{k!(-1)^{n+1}}{(k-n)!\,n!} A^{k-n} B A^{n+1}$$

$$= \sum_{n=0}^{k} \frac{k!(-1)^n}{(k-n)!\,n!} \left(A^{k+1-n} B A^n - A^{k-n} B A^{n+1}\right)$$

$$= \sum_{n=0}^{k} \frac{(-1)^n k!}{(k-n)!\,n!} A^{k+1-n} B A^n + \sum_{n=0}^{k} \frac{(-1)^{n+1} k!}{(k-n)!\,n!} A^{k-n} B A^{n+1}$$

Reindex second summation by putting $n' = n+1$

$$= \sum_{n=0}^{k} \frac{(-1)^n k!}{(k-n)!\,n!} A^{k+1-n} B A^n + \sum_{n'=1}^{k+1} \frac{(-1)^{n'} k!}{(k+1-n')!(n'-1)!} A^{k+1-n'} B A^{n'}$$

Put $n = n'$

$$= \sum_{n=0}^{k} \frac{(-1)^n k!}{(k-n)!\,n!} A^{k+1-n} B A^n + \sum_{n=1}^{k+1} \frac{(-1)^n k!}{(k+1-n)!(n-1)!} A^{k+1-n} B A^n$$

$$= A^{k+1}B + (-1)^{k+1} B A^{k+1} + \sum_{n=1}^{k} (-1)^n k! A^{k+1-n} B A^n \left(\frac{1}{(k-n)!\,n!} + \frac{1}{(k+1-n)!(n-1)!} \right)$$

$$= A^{k+1}B + (-1)^{k+1} B A^{k+1} + \sum_{n=1}^{k} \frac{(-1)^n k! A^{k+1-n} B A^n}{(k+1-n)!\,n!} \left(k+1-n + n\right)$$

$$= A^{k+1}B + (-1)^{k+1} B A^{k+1} + \sum_{n=1}^{k} \frac{(-1)^n (k+1)!}{(k+1-n)!\,n!} A^{k+1-n} B A^n$$

$$= \sum_{n=0}^{k+1} \frac{(-1)^n (k+1)!}{(k+1-n)!\,n!} A^{k+1-n} B A^n$$