

Question 5 - 10 marks

(a) Show that

$$\gamma: t \mapsto \begin{bmatrix} e^{\alpha t} \cos \beta t & e^{\alpha t} \sin \beta t \\ -e^{\alpha t} \sin \beta t & e^{\alpha t} \cos \beta t \end{bmatrix} \quad (\alpha, \beta \in \mathbb{R})$$

is a curve in $GL(2, \mathbb{R})$.

(b) Find the tangent vector v to γ at $\gamma(0)$.

(c) Show that

$$\begin{aligned} \phi: GL(2, \mathbb{R}) &\longrightarrow GL(1, \mathbb{R}) \\ A &\longmapsto \det A \end{aligned}$$

is a smooth homomorphism of matrix groups.

(d) Evaluate $d\phi(v)$ for the tangent vector v of part (b).

(e) Generalize this result.

[That is, show that $d\det = \text{Tr}$, where \det is the smooth homomorphism

$$\begin{aligned} \det: GL(n, \mathbb{R}) &\longrightarrow GL(1, \mathbb{R}) \\ A &\longmapsto \det A \end{aligned}$$

and where, for a matrix $A = (a_{ij})$,

$$\text{Tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}.]$$

TMA M824 02

Cut-off date 10 May 2000

Please make sure that the assignment number is correctly entered on your PT3 form as

M824 02

This TMA is based on Block II of the course.

Question 1 - 10 marks

(i) Define

$$B(t) = e^{tA} B e^{-tA} \quad (A, B \in M_n(\mathbb{R})).$$

Show that

$$B'(t) = [A, B(t)],$$

where $[A, B] \equiv AB - BA$.

[Differentiation of a matrix $B(t)$ is defined termwise; you may assume the obvious rules for differentiation here.]

(ii) By expanding

$$B(t) = \sum_{m=0}^{\infty} \frac{t^m}{m!} B_m,$$

show that

$$B_m = [A, [A, [A, \dots [A, B]]] \dots]$$

(where there are m A s in the square brackets).

(iii) Show that if $[A, [A, B]] = 0$ and $[B, [A, B]] = 0$, then

$$e^A e^B = e^B e^A e^{[A, B]}.$$

[This is a special case of the *Campbell-Baker-Hausdorff Formula* for Lie algebras.]