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$$= \frac{1}{2} Tr \begin{pmatrix} x_1 - ix_2 & x_0 + x_3 \\ x_0 - x_3 & x_1 + ix_2 \end{pmatrix}$$

$$= x_1$$

$$x_2 = \frac{Tr}{2}\left(\begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right)$$

$$= \frac{Tr}{2}\begin{pmatrix} ix_1 + x_2 & -ix_0 - ix_3 \\ ix_0 - ix_3 & -ix_1 + x_2 \end{pmatrix} = x_2$$

$$x_3 = \frac{Tr}{2}\left(\begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$

$$= \frac{Tr}{2}\begin{pmatrix} x_0 + x_3 & -x_1 + ix_2 \\ x_1 + ix_2 & -x_0 + x_3 \end{pmatrix}$$

$$= x_3$$

$\therefore x_\mu = \frac{1}{2} Tr(X \tau_\mu)$ in each case. ✓

iii) $X = \begin{bmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{bmatrix}$

$$\det \begin{bmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{bmatrix}$$

$$= (x_0 + x_3)(x_0 - x_3) - (x_1 - ix_2)(x_1 + ix_2)$$

$$= x_0^2 - x_3^2 - (x_1^2 + \cancel{ix_1x_2} - \cancel{ix_2x_1} + x_2^2)$$

$$= x_0^2 - x_3^2 - x_1^2 - x_2^2.$$

$$\det(gXg^+) = \det(g)\det(X)\det(g^+)$$

$$= 1 \times \det(X) \det(\bar{{}^tg})$$

$$= \det(X) \overline{\det({}^tg)}$$

$$= \det(X) \times 1 = \det X$$

since $\det({}^tg) = \det(g) = 1$.

$\therefore x_0^2 - x_1^2 - x_2^2 - x_3^2$ is invariant under $X \to gXg^+$, so it is a Lorentz transformation.

Needs linearity too $-\frac{1}{2}$