(4)

2) i) Any 2×2 Hermitian matrix takes the form $\begin{bmatrix} a & b_1+ic_1 \\ b_1-ic_1 & d_1 \end{bmatrix}$

$= \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & ic \\ -ic & 0 \end{bmatrix}$

$= \frac{(a-d)}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{(a+d)}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - c\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$= \frac{(a-d)}{2}\tau_3 + \frac{(a+d)}{2}\tau_0 + b\tau_1 - c\tau_2$

$\therefore \{\tau_0, \tau_1, \tau_2, \tau_3\}$ span X. ✓

Suppose $a\tau_0 + b\tau_1 + c\tau_2 + d\tau_3 = 0$. then

$a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + d\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$= \begin{bmatrix} a+d & b-ic \\ b+ic & a-d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore a+d = 0$
$a-d = 0 \Rightarrow a=d=0$

$b-ic=0$
$b+ic=0 \Rightarrow b=c=0$

$\therefore \{\tau_0, \tau_1, \tau_2, \tau_3\}$ is a linearly independent set, so is a basis for X. ✓

ii) $X = \begin{bmatrix} x_0+x_3 & x_1-ix_2 \\ x_1+ix_2 & x_0-x_3 \end{bmatrix}$

$x_0 = \frac{\mathrm{Tr}}{2}\left(\begin{pmatrix} x_0+x_3 & x_1-ix_2 \\ x_1+ix_2 & x_0-x_3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$

$= \frac{1}{2}(x_0+x_3+x_0-x_3) = x_0$

$x_1 = \frac{\mathrm{Tr}}{2}\left(\begin{pmatrix} x_0+x_3 & x_1-ix_2 \\ x_1+ix_2 & x_0-x_3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$