Question 4 – 10 marks

Consider the surjective homomorphism defined on page 61 of Curtis by

$$\rho : S^3 \longrightarrow SO(3).$$

(i) Show that ρ is smooth.

(ii) Find $d\rho : T_{S^3} \longrightarrow T_{SO(3)}$, i.e. show that $d\rho(\gamma'(0))$ is the transformation

$$(r_1, r_2, r_3) \longmapsto (r_1, r_2, r_3)\, 2 \begin{bmatrix} 0 & \lambda & -\nu \\ -\lambda & 0 & \mu \\ \nu & -\mu & 0 \end{bmatrix},$$

where

$$\gamma'(0) = i\mu + j\nu + k\lambda \in T_{S^3}.$$

Question 5 – 10 marks

Let $SU(m,n)$ be the set of $(m+n)\times(m+n)$ complex matrices u of determinant 1 satisfying $uKu^\dagger = K$, where

$$K = \begin{bmatrix} I_m & 0 \\ 0 & -I_n \end{bmatrix},$$

where I_m and I_n are the $m \times m$ and $n \times n$ unit matrices respectively, and where $u^\dagger \equiv {}^t\overline{u}$ is the Hermitian conjugate of u.

$uK = Ku^{\dagger}$

(i) Show that $SU(m,n)$ is a group.

(ii) If A is a tangent vector to $SU(m,n)$, show that $\mathrm{Tr}(A) = 0$ and that

$$AK + KA^\dagger = 0.$$

$AK = KA$

(iii) Show that $su(m,n)$, the set of all matrices A satisfying the relations in part (ii) above, is a real vector space, and find its dimension.

(iv) Show that $su(m,n)$ is a Lie algebra, i.e. that $su(m,n)$ is closed under the operation

$$[A,B] = AB - BA \qquad (A, B \in su(m,n)).$$

(v) Describe the vector space $su(m,n) \cap su(m+n)$, and find its dimension.
