Question 2 – 10 marks

(i) Show that the set of matrices $\{\tau_0, \tau_1, \tau_2, \tau_3\}$, where

$$\tau_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tau_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \tau_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \tau_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

is a basis for the vector space (over $\mathbb{R}$) of all 2×2 Hermitian matrices X.

(ii) Show that every Hermitian matrix X determines a 4-vector x_μ via

$$x_\mu = \tfrac{1}{2}\mathrm{Tr}(X\tau_\mu).$$

(iii) In special relativity, a *Lorentz transformation* is defined to be a linear mapping of a 4-vector x_μ that preserves $x_0^2 - x_1^2 - x_2^2 - x_3^2$.

If $g \in SL(2,\mathbb{C})$, show that $X \longmapsto gXg^\dagger$, where $g^\dagger = {}^t\overline{g}$ is the Hermitian conjugate of g, defines a Lorentz transformation of the 4-vector x_μ.

[$SL(2,\mathbb{C}) = \{x \in M_2(\mathbb{C}) \mid \det x = 1\}$.]

(iv) Show that the map $X \longmapsto uXu^\dagger$, $u \in SU(2)$, where $u^\dagger = {}^t\overline{u}$ is the Hermitian conjugate of u, defines an orthogonal transformation of (x_1, x_2, x_3) in 3-space.

(v) Show that the map in part (iv) is a *rotation* in 3-space.

[You may assume that every reflection in 3-space is of the form $(x_1 \longmapsto -x_1, x_2 \longmapsto x_2, x_3 \longmapsto x_3)$ times a rotation.]

Question 3 – 10 marks

Let V be a finite-dimensional vector space, and let L be a Lie algebra (which is itself a finite-dimensional vector space).

An *endomorphism* of V is a linear transformation from V into V. Denote the set of endomorphisms of V by $\mathrm{End}\, V$. $\mathrm{End}\, V$ is a vector space.

A *representation* of L is a linear transformation ϕ from L to $\mathrm{End}\, V$ such that

$$\phi[X,Y] = [\phi(X), \phi(Y)],$$

where $[X,Y] \equiv XY - YX$ and $[\phi(X), \phi(Y)] \equiv \phi(X)\phi(Y) - \phi(Y)\phi(X)$.

(i) Show that for a fixed $g \in L$, the map

$$\begin{aligned} \mathrm{ad}\, g : L &\longrightarrow L \\ x &\longmapsto \mathrm{ad}\, g(x) \equiv [g, x] \end{aligned}$$

is an endomorphism of L.

(ii) Show that the map

$$\begin{aligned} \mathrm{ad} : L &\longrightarrow \mathrm{End}\, L \\ x &\longmapsto \mathrm{ad}\, x \end{aligned}$$

is a representation of L.

[This representation is called the *adjoint* representation of L.]

(iii) Find the adjoint representation of the Lie algebra $sp(1)$.

[That is, for any element $g \in sp(1)$, find a matrix representing the linear transformation $\mathrm{ad}\, g$ on the real vector space $sp(1)$ with respect to a chosen basis.]