Question 3 – 10 marks

Let $G \subset M_n(\mathbb{C})$ be a matrix group.

(i) Show that each of the following maps defines an isomorphism from G to its image.

$$\phi_1 : g \longrightarrow \overline{g}$$
$$\phi_2 : g \longmapsto {}^t g^{-1}$$
$$\phi_3 : g \longmapsto (g^\dagger)^{-1}$$

[$g^\dagger \equiv {}^t\overline{g}$ is the Hermitian conjugate of g.]

(ii) Show that the map

$$\phi : G \longrightarrow G' = \phi(G)$$
$$g \longmapsto TgT^{-1},$$

where T is an invertible element of $M_n(\mathbb{C})$, defines an isomorphism.

[The matrix groups G and G' are then said to be *equivalent*.]

(iii) An isomorphism $\phi : G \longrightarrow G$ is said to be an *automorphism*.

Show that when $T \in G$, ϕ as defined in part (ii) above is an automorphism.

[Such a ϕ is called an *inner* automorphism.]

(iv) Consider the case $G \equiv U(n)$ and the isomorphisms defined in part (i) above.

(a) Show that ϕ_3 is the identity inner automorphism.

(b) Show that $\phi_1 = \phi_2$.

(c) Show that ϕ_1 is not an inner automorphism.

[An automorphism that is not an inner automorphism is called an *outer* automorphism.]

(v) Consider the set $\mathcal{A}(G)$ of all automorphisms of the matrix group G.

(a) Show that $\mathcal{A}(G)$ is a group, using composition of functions as the group operation.

[The identity is the identity automorphism.]

(b) Show that $\mathcal{I}(G)$, the set of inner automorphisms of G, is a normal subgroup of $\mathcal{A}(G)$.

(c) Show that $\mathcal{I}(G)$ is isomorphic to $G/\text{Centre}(G)$.

m

k

Question 4 – 10 marks

$\kappa(g) = a_1 a_2 g a_3 a_4$

Let M be a fixed matrix in $M_n(\mathbb{C})$.

$g^{-1} = a_2^{-1} a_1^{-1} g a_4^{-1} a_3^{-1}$

Define a bilinear form $\{x, y\}$ on $\mathbb{C}^n \times \mathbb{C}^n$ by

$$\{x, y\} = x M y^\dagger,$$

where $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ and $y^\dagger = {}^t\overline{y}$ is the Hermitian conjugate of the vector y.

Find necessary and sufficient conditions on the matrix M for which *each* of the properties (i)–(vii) of Proposition 1 of Chapter 2 of Curtis (page 24) holds for this bilinear form. Treat *each* property in isolation.

[You may assume standard results about complex matrices, including the fact that every Hermitian matrix may be diagonalized by a unitary matrix.]

Can't prove monogenic of [0,1] like this: let $x = 0.10110121\ldots$ then the decimal expansion of x takes on every sequence so $|10^k x^{\text{mod}1} - 9| < \varepsilon$ for any ε and some k.