Please make sure that the assignment number is correctly entered on your PT3 form as

$\boxed{\text{M824 01}}$.

This TMA is based on Block I of the course.

Question 1 – 10 marks

Show that the map

$$\begin{aligned} \phi_0 : GL(n, \mathbb{C}) &\longrightarrow GL(2, \mathbb{C}) \\ A &\longmapsto \begin{bmatrix} \det A & 0 \\ 0 & (\overline{\det A}) \end{bmatrix} \end{aligned}$$

(where $(\overline{\det A})$ is the complex conjugate of the determinant of A) is a group homomorphism.

Show that ϕ_0 induces a homomorphism

$$\phi : U(n) \longrightarrow SU(2).$$

Describe the standard matrix groups which are isomorphic to $\operatorname{Ker} \phi$ and $\operatorname{Im} \phi$.

Using the notation $C(G)$ for the centre of the group G, show that

$$C(SU(2)) \subset C(\operatorname{Im} \phi).$$

[$\operatorname{Ker} \phi = \{x \in U(n) \mid \phi(x) = I\}$,
$\operatorname{Im} \phi = \phi(U(n)) = \{y \in SU(2) \mid \exists x \in U(n) \text{ such that } \phi(x) = y\}$.]

Question 2 – 10 marks

(i) Show that if the real square matrix A is skew-symmetric, then $\det(I + A)$ and $\det(I - A)$ are both non-zero, where I is the unit matrix.

(ii) With A as in part (i) above, show that $(I - A)(I + A)^{-1}$ exists and is orthogonal.

(iii) Conversely, show that if the matrix Q is orthogonal, and $\det(I + Q) \neq 0$, then $(I - Q)(I + Q)^{-1}$ is skew-symmetric.

[This relation between skew-symmetric and orthogonal sets of matrices is due to the English mathematician Cayley (1821–1895).]

$B^T B = I$