$\det A = \sum (-1)^\delta a_{1i_1} a_{2i_2} \cdots a_{ni_n}$
where δ is the number of transpositions required to restore $i_1, i_2, \ldots, i_n$ to their natural order.

$(\det A)'(0) = \sum (-1)^\delta a'_{1i_1} a_{2i_2} \cdots a_{ni_n}$
$+ \cdots + \sum (-1)^\delta a_{1i_1} a_{2i_2} \cdots a'_{ni_n}$
evaluated at $t=0$.
$a_{ij} = 0$ if $i \neq j$, 1 if $i=j$, therefore each sum of the form
$\sum (-1)^\delta a_{1i_1} \cdots a'_{pi_p} \cdots a_{ni_n} = 0$ if all the elements except the p^{th} are not diagonal, but then the p^{th} is diagonal and $\delta = 0$

$\therefore (\det A)'(0) =$ Sum of all diagonal elements whose differential is evaluated at 0, i.e. $(\det A)'(0) = (\mathrm{Tr} A)'(0)$