P6578708 (2)

$\sum_{r=1}^{70} r^2(r|71) = \frac{1}{2}(5041\sum_{r=1}^{70}(71-r|71) - 142\sum_{r=1}^{70} r(71-r|71))$ ✓

$= \frac{1}{2}(0 - 142\sum_{r=1}^{70} r(71-r|71)) = \frac{1}{2}(-142\sum_{r=1}^{70} -r(r|71))$ ✓

$= 71\sum_{r=1}^{70} r(r|71)$. as required. ✓ 2½

ii) Suppose there are only finitely many primes of the form $8k-1$, $p_1 \ldots p_n$ say. Consider the number ✓

$N = 16p_1^2 p_2^2 \ldots p_n^2 - 2$.

Let p be any prime dividing N, then $N \equiv 0 \pmod{p}$ ✓

ie $16\prod_{i=1}^{n} p_i^2 - 2 \equiv 0 \pmod{p}$ ✓

$16\prod_{i=1}^{n} p_i^2 \equiv 2 \pmod{p}$ ✓

$\therefore$ 2 is a quadratic residue of p, so $p \equiv 1$ or $7 \pmod{8}$. p odd, so $p \neq 2$

$\therefore N/2 = 8\prod_{i=1}^{n} p_i^2 - 1 \equiv 0 \pmod{p}$. If the only primes dividing N (besides 2) were of the form $8k+1$ then $N/2$ would be of this form since $(8n+1)(8m+1) = 8(8nm+n+m)+1$ $\therefore$ a prime of form $8k+7$ divides N ✓

$\therefore p|N$, $p|\prod_{i=1}^{n} p_1^2 p_2^2 \ldots p_n^2$, since $p = p_i$ for some i ✓

$\Rightarrow p | N - \prod_{i=1}^{n} p_i^2 \Rightarrow p | 2$, which is a contradiction since p, 2 are prime and $p \neq 2$ $\therefore$ infinitely many primes of form $8k-1$. ✓ 4

iii) The sets S and T are defined so that

1. $x, y \in S \Rightarrow x \cdot y \in S$
2. $x, y \in T \Rightarrow x \cdot y \in S$
3. $x \in S$, $y \in T \Rightarrow x \cdot y \in T$ ✓