1c

$$\theta_1(x+\varepsilon x)-\theta_1(x)=\int_x^{x+\varepsilon x}\theta(t)\,dt \geq (x+\varepsilon x-x)\theta(x)$$

$$\geq \varepsilon x\,\theta(x) \quad (a)$$

Since θ_1 increasing.

Since $\lim_{x\to\infty}\frac{\theta_1(x)}{\frac{1}{2}x^2}=l$ there exists ε_1, x_0

such that $\left|\frac{\theta_1(x)}{\frac{1}{2}x^2}-l\right|<\varepsilon_1$ for all $x>x_0$

ie $$-\varepsilon_1<\frac{\theta_1(x)}{\frac{1}{2}(x^2)}-l<\varepsilon_1$$

$$(l-\varepsilon_1)\frac{x^2}{2}<\theta_1(x)<(l+\varepsilon_1)\frac{x^2}{2} \quad (1)$$

Since $x+\varepsilon x>x>x_0$ we have a similar expression for $\theta_1(x+\varepsilon x)$

$$(l-\varepsilon_1)\frac{(x+\varepsilon x)^2}{2}<\theta_1(x+\varepsilon x)<(l+\varepsilon_1)\frac{(x+\varepsilon x)^2}{2} \quad (1)^*$$

Also, since $\lim_{x\to\infty}\frac{\theta(x)}{x}=1$, there exists x_1 such that $\left|\frac{\theta(x)}{x}-1\right|<\varepsilon_1$ for all $x>x_1$ ie $-\varepsilon_1<\frac{\theta(x)}{x}-1<\varepsilon_1$

$$x(1-\varepsilon_1)<\theta(x)<x(1+\varepsilon_1) \quad (2)$$

Choose $x_2=\mathrm{Max}\{x_0,x_1\}$

Then (1), (2) hold with this value, $x>x_2$.

Now use (1), (1)* and (2) in (a)

$$(l+\varepsilon_1)\frac{(x+\varepsilon x)^2}{2}-(l-\varepsilon_1)\frac{x^2}{2}>\theta_1(x+\varepsilon x)-\theta_1(x)\geq\varepsilon x\,\theta(x$$

$$\frac{lx^2}{2}\left((1+\varepsilon)^2-1\right)+\frac{\varepsilon_1 x^2}{2}\left((1+\varepsilon)^2+1\right)>\varepsilon x^2(1-\varepsilon_1)$$