TMA 02 M823.

1) i) Euler's summation formula states

$$\sum_{y<n\le x} f(n) = \int_y^x f(t)\,dt + \int_y^x (t-[t])f'(t)\,dt - (x-[x])f(x) + (y-[y])f(y). \checkmark$$

$$\sum_{n\le x}\frac{\log n}{n^3} = \frac{\log 1}{1^3} + \sum_{1<n\le x}\frac{\log n}{n^3} = \sum_{1<n\le n^3}\frac{\log n}{n^3}$$

$$\sum_{1<n\le x}\frac{\log n}{n^3} = \int_1^x \frac{\log t}{t^3}\,dt + \int_1^x (t-[t])\left(\frac{1}{t^4} - \frac{3\log t}{t^4}\right)dt - (x-[x])\frac{\log x}{x^3} + (1-[1])\frac{\log 1}{1^3} \checkmark$$

$$= \int_1^x \frac{\log t}{t^3}\,dt + \int_1^x \frac{(t-[t])(1-3\log t)}{t^4}\,dt + O\left(\frac{\log x}{x^3}\right) \checkmark$$

since last term is zero and $(x-[x]) = O(1)$.

$$\int_1^x (t-[t])\left(\frac{1}{t^4} - \frac{3\log t}{t^4}\right)dt = K + O\left(1\int_1^x \frac{1}{t^4} - \frac{3\log t}{t^4}\,dt\right) \checkmark$$

$$= O\left(\left[\frac{\log t}{t^3}\right]_1^x\right) = O\left(\frac{\log x}{x^3}\right) + K$$

PTO.

$$\therefore \sum_{1<n\le x}\frac{\log n}{n^3} = \int_1^x \frac{\log t}{t^3}\,dt + K + O\left(\frac{\log x}{x^3}\right) \checkmark$$

We integrate by parts.

$u = \log t \Rightarrow du = 1/t$

$dv = t^{-3} \Rightarrow v = -\frac{1}{2}t^{-2}$

$$\int_1^x \frac{\log t}{t^3}\,dt = \left[-\frac{1}{2}t^{-2}\log t\right]_1^x - \int_1^x \frac{1}{t}\times -\frac{1}{2}t^{-2}\,dt \checkmark$$