Please make sure that the assignment number is correctly entered on your PT3 form as

$\boxed{\text{M823 02}}$.

Question 1 (on Chapter 3)

(i) Use Euler's summation formula to prove that for $x \geq 2$,

$$\sum_{n \leq x} \frac{\log n}{n^3} = A - \frac{\log x}{2x^2} - \frac{1}{4x^2} + O\left(\frac{\log x}{x^3}\right),$$

where A is a constant. [6]

(ii) Prove that for $x \geq 2$,

$$\sum_{n \leq x} \frac{d(n)}{n^4} = B - \frac{\log x}{3x^2} + O(x^{-3}),$$

where B is a constant which you should determine. [6]

(iii) Prove that for $x \geq 2$,

$$\sum_{n \leq x} \frac{\phi(n)}{n^{1/4}} = Cx^{7/4} + O\left(x^{3/4} \log x\right),$$

where C is a constant which you should determine. [6]

(iv) Find the highest power of 7 dividing 2000!. [2]

Question 2 (on Chapter 4)

(i) Define the logarithmic integral $li(x)$ by

$$li(x) = \int_2^x \frac{dt}{\log t}, \quad \text{for } x > 2.$$

(a) Prove that

$$li(x) = \frac{x}{\log x} + \int_2^x \frac{dt}{\log^2 t} + A \qquad (*)$$

and

$$li(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + 2\int_2^x \frac{dt}{\log^3 t} + B,$$

for some constants A and B which you should determine.

(b) Use equation $(*)$ above to prove that $li(x) \sim x/\log x$; deduce that the Prime Number Theorem can be expressed in the form $\pi(x) \sim li(x)$. [8]