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$$= \frac{x}{\log x} + \frac{x}{\log^2 x} + 2\int_2^x \frac{dt}{\log^3 t} - \left(\frac{2}{\log 2} + \frac{2}{\log^2 2}\right) \checkmark$$

4.

b) $$li(x) = \frac{x}{\log x} + \int_2^x \frac{dt}{\log^2 t} + A \checkmark$$

$$\frac{li(x)\log x}{x} = 1 + \frac{\log x}{x}\int_2^x \frac{dt}{\log^2 t} + \frac{A\log x}{x} \checkmark$$

as $x \to \infty$, $\frac{\log x}{x} \to 0 \Rightarrow \frac{A\log x}{x} \to 0$ ✓

$$0 < \frac{\log x}{x}\int_2^x \frac{dt}{\log^2 t} \leq \frac{\log x}{x}\left(\frac{\sqrt{x}-2}{\log^2 2} + \frac{x-\sqrt{x}}{\log^2 \sqrt{x}}\right) \checkmark$$

$$< \frac{\log x}{x} \times \frac{\sqrt{x}}{\log^2 2} + \frac{\log x}{x} \times \frac{x}{\frac{1}{4}\log^2 x} \checkmark$$

$$= \frac{\log x}{\sqrt{x}\log^2 2} + \frac{4}{\log x} \checkmark$$

and this expression also tends to zero as $x \to \infty$. ✓

$$\therefore \lim_{x\to\infty} \frac{li(x)\log x}{x} = 1 \checkmark$$

$$\Rightarrow li(x) \sim \frac{x}{\log x}. \checkmark$$

The prime number theorem states

$$\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1 \checkmark$$

$$\therefore \pi(x) \sim \frac{x}{\log(x)} \sim li(x) \checkmark$$

$$\therefore \pi(x) \sim li(x). \checkmark$$

4

Good.