$$(x-\varepsilon x) < x \Rightarrow \frac{1}{(x-\varepsilon x)} > \frac{1}{x}$$

and $\frac{1}{(x-\varepsilon x)} < \frac{1}{x}$

$$\therefore \frac{\theta_1(x)}{x^2} - \frac{\theta_1(x-\varepsilon x)}{(x-\varepsilon x)^2} \le \frac{\theta_1(x)}{x^2} - \frac{\theta_1(x-\varepsilon x)}{x^2} < \frac{\varepsilon\theta(x)}{x}$$ ✓

Now let $x \to \infty$. For ε given we can choose x large enough such that, for some small ε_1, why? pto.

$$\frac{\theta_1(x)}{\frac{1}{2}x^2} > l-\varepsilon_1, \quad \frac{\theta_1(x-\varepsilon x)}{\frac{1}{2}(x-\varepsilon x)^2} < l+\varepsilon_1, \quad \frac{\theta}{x} < 1+\varepsilon_1$$

$$\text{so } \theta_1(x) > \frac{x^2}{2}(l-\varepsilon_1), \quad \theta_1(x-\varepsilon x) < \frac{x^2(1-\varepsilon)^2(l+\varepsilon_1)}{2}$$

$$\frac{x^2(l-\varepsilon_1)}{2} - \frac{x^2(1-\varepsilon)^2(l+\varepsilon_1)}{2} < \theta_1(x) - \theta_1(x-\varepsilon x) < \varepsilon x^2(1+\varepsilon_1)$$

$$\text{ie } \frac{(l-\varepsilon_1)}{2} - \frac{(1-\varepsilon)^2(l+\varepsilon_1)}{2} < \varepsilon(1+\varepsilon_1)$$

$$\varepsilon_1\left(\frac{-1}{2} - \frac{(1-\varepsilon)^2}{2}\right) + \frac{l}{2}\left(1-(1-\varepsilon)^2\right) < \varepsilon + \varepsilon\varepsilon_1$$

$$\varepsilon_1\left(-\frac{1}{2} - \frac{1}{2} + \varepsilon - \frac{\varepsilon^2}{2} - \varepsilon\right) + \frac{l}{2}\left(2\varepsilon - \varepsilon^2\right) < \varepsilon$$

$$\varepsilon_1\left(-1 - \varepsilon^2/2\right) + l\varepsilon(1-\varepsilon/2) < \varepsilon$$

$$\text{ie } \frac{\varepsilon_1}{\varepsilon}\left(-1 - \frac{\varepsilon^2}{2}\right) + l(1-\varepsilon/2) < 1$$

ε is given at this stage so we can choose values of x so large that we can let $\varepsilon_1/\varepsilon \to 0$. then the above equation will still be true, since it is true for any $\varepsilon_1 > 0$. We can therefore write $l(1-\varepsilon/2) \le 1$.

(We no longer write $<$ since we have let $\varepsilon_1/\varepsilon \to 0$)