(ii) Let $\theta_1(x) = \int_1^x \theta(t)\,dt$, for $x > 1$.

(a) Calculate $\theta_1(6\frac{1}{2})$ using the above definition.

(b) Let $\Lambda_1(n) = \log n$ if n is prime, and 0 otherwise. Prove that

$$\theta_1(x) = \sum_{n \le x} (x - n)\Lambda_1(n),$$

and hence check your answer to part (a).

(c) Let $l = \lim_{x\to\infty} \theta_1(x)/\frac{1}{2}x^2$.

By considering $\int_{x-\varepsilon x}^{x} \theta(t)\,dt$, prove that

$$\theta_1(x) - \theta_1(x - \varepsilon x) \le \varepsilon x \cdot \theta(x), \qquad \text{where } \varepsilon\ (> 0) \text{ is small.}$$

Assuming that $\theta(x)/x \to 1$, deduce that $(1 - \frac{1}{2}\varepsilon)l \le 1$.

By similarly considering $\int_{x}^{x+\varepsilon x} \theta(t)\,dt$, prove that $(1 + \frac{1}{2}\varepsilon)l \ge 1$.

Deduce that $\theta_1(x) \sim \frac{1}{2}x^2$. *[12]*

Question 3 (on Chapter 5)

(i) Find the remainder when 2000^{2000} is divided by 13. [4]

(ii) Prove that

$$e^x \cdot e^{x^2/2} \cdot e^{x^3/3} \cdots = 1 + x + x^2 + \cdots.$$

Show that the coefficient of x^{17} in the power series expansion of the left-hand side has the form

$$1/17! + 1/17 + r/s,$$

where 17 does not divide s. Deduce that

$$16! \equiv -1 \pmod{17}.$$ [6]

(iii) Let

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{823} = \frac{r}{823s}.$$

Without calculating the left-hand side, prove that $r \equiv s \pmod{823^3}$. [4]

(iv) Find both solutions of the polynomial congruence

$$3x^2 + 2x + 100 \equiv 0 \pmod{343}.$$ [6]

$\frac{r}{s}$ $\frac{1}{823}\left(\frac{r}{s} - 1\right) = \frac{1}{823}\left(\frac{r-s}{s}\right)$