M821: TMA03 2001
Solution for question 2

Part i

Suppose that the solution of the equation can be written in the form

$$\begin{aligned} y(x,\epsilon) &= Y(x,\eta,\epsilon), \quad \eta = x/\epsilon \\ &= Y_0(x,\eta) + \epsilon Y_1(x,\eta) + O(\epsilon^2). \end{aligned} \tag{1}$$

Then, since

$$\begin{aligned} \frac{dy}{dx} &= \frac{\partial Y}{\partial x} + \frac{1}{\epsilon}\frac{\partial Y}{\partial \eta} \quad \text{and} \\ \frac{d^2y}{dx^2} &= \frac{\partial^2 Y}{\partial x^2} + \frac{2}{\epsilon}\frac{\partial^2 Y}{\partial x \partial \eta} + \frac{1}{\epsilon^2}\frac{\partial^2 Y}{\partial \eta^2}, \end{aligned}$$

the original differential equation can be written as

$$\frac{1}{\epsilon}\left(\frac{\partial^2 Y}{\partial \eta^2} + \frac{\partial Y}{\partial \eta}\right) + \left(2\frac{\partial^2 Y}{\partial x \partial \eta} + \frac{\partial Y}{\partial x} + x^2 Y^2\right) + \epsilon\frac{\partial^2 Y}{\partial x^2} = 0.$$

On substituting the expansion 1 into this equation gives

$$\frac{1}{\epsilon}\left(\frac{\partial^2 Y_0}{\partial \eta^2} + \frac{\partial Y_0}{\partial \eta}\right) + \left(\frac{\partial^2 Y_1}{\partial \eta^2} + \frac{\partial Y_1}{\partial \eta} + 2\frac{\partial^2 Y_0}{\partial x \partial \eta} + \frac{\partial Y_0}{\partial x} + x^2 Y_0^2\right) = O(\epsilon). \tag{2}$$

Equating the coefficient of ϵ to zero gives the following equation for Y_0

$$\frac{\partial^2 Y_0}{\partial \eta^2} + \frac{\partial Y_0}{\partial \eta} = 0. \tag{3}$$

Part ii

Equation 3 can be re-written in the form

$$\frac{\partial}{\partial \eta}\left(e^{\eta}\frac{\partial Y_0}{\partial \eta}\right) = 0 \quad \text{so} \quad Y_0(x,\eta) = A(x) + B(x)e^{-\eta},$$

where $A(x)$ and $B(x)$ are arbitrary function yet to be found.

Equating to zero the coefficient of ϵ^0 in equation 2 gives

$$\frac{\partial^2 Y_1}{\partial \eta^2} + \frac{\partial Y_1}{\partial \eta} + 2\frac{\partial^2 Y_0}{\partial x \partial \eta} + \frac{\partial Y_0}{\partial x} + x^2 Y_0^2 = 0. \tag{4}$$

But

$$\begin{aligned} \frac{\partial Y_0}{\partial x} &= A'(x) + B'(x)e^{-\eta}, \quad \text{and} \\ \frac{\partial^2 Y_0}{\partial x \partial \eta} &= -B'(x)e^{-\eta} \end{aligned}$$