Question 2 – 25 marks

(a) Use the method of multiple scales to show that the solution of the differential equation

$$\epsilon \frac{d^2y}{dx^2} + \frac{dy}{dx} + (xy)^2 = 0, \quad y(\epsilon, 0) = 0,$$

can be written in the form

$$y(\epsilon, x) = Y_0(x, \eta) + \epsilon Y_1(x, \eta) + O(\epsilon^2)$$

where $\eta = x/\epsilon$, and that the function Y_0 satisfies the equation

$$\frac{\partial^2 Y_0}{\partial \eta^2} + \frac{\partial Y_0}{\partial \eta} = 0.$$

[5]

(b) By solving this equation show that

$$Y_0(x, \eta) = A(x) + B(x)e^{-\eta}$$

where $A(x)$ and $B(x)$ are functions of x only. Hence show that the equation satisfied by A, B and Y_1 is

$$\frac{\partial^2 Y_1}{\partial \eta^2} + \frac{\partial Y_1}{\partial \eta} + (xB)^2 e^{-2\eta} + \left(2x^2AB - \frac{dB}{dx}\right)e^{-\eta} + \frac{dA}{dx} + (xA)^2 = 0.$$

[5]

(c) By making suitable assumptions about Y_1 derive three separate equations for A, B and Y_1 and deduce that $B(x) = c/A(x)^2$ for some constant c, and that A and Y_1 are given by the solutions of the equations

$$\frac{dA}{dx} = -(xA)^2, \quad \frac{\partial^2 Y_1}{\partial \eta^2} + \frac{\partial Y_1}{\partial \eta} = -x^2B^2e^{-2\eta}.$$

[10]

(d) Hence show that an approximate solution to the original equation is

$$y(x, \epsilon) \simeq \frac{3}{\alpha + x^3} - \frac{3}{\alpha^3}(\alpha + x^3)^2 e^{-x/\epsilon} - \frac{9\epsilon x^2}{2\alpha^6}(\alpha + x^3)^4 e^{-2x/\epsilon},$$

where α is a constant.

[5]

Question 3 – 25 marks

Consider the modified van der Pol equation,

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x + \epsilon\alpha^2x^3 = 0; \quad x(0) = a > 0, \quad \dot{x}(0) = 0, \quad |\epsilon| \ll 1,$$

where a and α are positive constants.

(a) Assuming that a periodic solution exists, use the method of Lindstedt to show that it may be approximated by the series

$$x(\tau) = x_0(\tau) + \epsilon x_1(\tau) + O(\epsilon^2), \quad \tau = \omega t,$$

where $\omega = 1 + \epsilon\omega_1 + O(\epsilon^2)$ and $x_0(0) = a$, $\dot{x}_0 = 0$ and $x_k(0) = \dot{x}_k(0) = 0$, $k = 1, 2, \cdots$. Deduce that

$$x_0(\tau) = a\cos\tau.$$

[5]

(b) Show that x_1 satisfies the equation

$$\begin{aligned} \frac{d^2x_1}{d\tau^2} + x_1 &= a\left(\tfrac{1}{4}a^2 - 1\right)\sin\tau + 2a\left(\omega_1 - \tfrac{3}{8}a^2\alpha^2\right)\cos\tau \\ &\quad + \tfrac{1}{4}a^3\sin 3\tau - \tfrac{1}{4}\alpha^2a^3\cos 3\tau \end{aligned}$$

and deduce that, for a periodic solution,

$$a = 2, \quad \omega_1 = \tfrac{3}{2}\alpha^2.$$

[10]

(c) Hence show that an approximate periodic solution is

$$x(t) = 2\cos\tau - \tfrac{1}{4}\epsilon\left\{3\sin\tau - \sin 3\tau - \alpha^2(\cos\tau - \cos 3\tau)\right\}$$

with $\tau = \omega t$, where $\omega = 1 + \frac{3}{2}\alpha^2\epsilon$.

[10]