M821: TMA03 2001
Solution for question 3

Assume that a periodic solution with, unknown, frequency ω exists and set $\tau = \omega t$ with $x(\tau)$ 2π-periodic in τ. The equation of motion becomes

$$\omega^2 \frac{d^2 x}{d\tau^2} + \omega\epsilon(x^2 - 1)\frac{dx}{d\tau} + x + \epsilon\alpha^2 x^3 = 0, \quad x(0) = a > 0, \quad x'(0) = 0.$$

In the unperturbed limit there are infinitely many periodic solutions all with unit frequency. The perturbation destroys all but one of these which we find by setting $\omega = 1 + \epsilon\omega_1 + \cdots$ and $x = x_0(\tau) + \epsilon x_1(\tau) + \cdots$. The zero-order equation is

$$\frac{d^2 x_0}{d\tau^2} + x_0 = 0.$$

Choosing the initial conditions to be $(x_0(0), x_0'(0)) = (a, 0)$ and $(x_k(0), x_k'(0) = 0)$ for $k \geq 1$, we obtain $x_0 = a\cos\tau$.

The first-order expansion of the equation of motion is

$$\begin{aligned}
\frac{d^2 x_1}{d\tau^2} + x_1 &= (1 - x_0^2)\frac{dx_0}{d\tau} - 2\omega_1 \frac{d^2 x_0}{d\tau^2} - \alpha x_0^3 \\
&= -a(1 - a^2\cos^2\tau)\sin\tau + 2a\omega_1\cos\tau - \frac{1}{4}\alpha^2 a^3 (3\cos\tau + \cos 3\tau) \\
&= \left(2a\omega_1 - \frac{3}{4}\alpha^2 a^3\right)\cos\tau + a\left(\frac{a^2}{4} - 1\right)\sin\tau + \frac{a^3}{4}\sin 3\tau - \frac{1}{4}\alpha^2 a^3 \cos 3\tau
\end{aligned}$$

This equation has a periodic solution only if the coefficients of $\sin\tau$ and $\cos\tau$ are both zero, that is

$$a = 2, \quad \text{and} \quad 2a\omega_1 = \frac{3}{4}\alpha^2 a^3 \implies \omega_1 = \frac{3}{2}\alpha^2$$

Hence the equation for x_1 becomes

$$\frac{d^2 x_1}{d\tau^2} + x_1 = \frac{a^3}{4}\sin 3\tau - \frac{\alpha^2 a^3}{4}\cos 3\tau$$

which, since $a = 2$, has the solution

$$x_1 = a_1\cos\tau + b_1\sin\tau - \frac{1}{4}\sin 3\tau + \frac{\alpha^2}{4}\cos 3\tau.$$

The initial conditions then give

$$x_1(0) = 0 = a_1 + \frac{\alpha^2}{4} \qquad x_1'(0) = b_1 - \frac{3}{4}.$$

Hence the first-order approximation to the periodic solution is

$$x(t) = 2\cos t + \frac{\epsilon}{4}(3\sin t - \sin 3t - \alpha^2(\cos\tau - \cos 3\tau)).$$