Please make sure that the assignment number is correctly entered on your PT3 form as

$$\boxed{\text{M821 01}}.$$

Question 1 – 10 marks

Classify the fixed points of the following linear systems and state whether they are stable or unstable.

(a) $\dot{x} = 3x + 2y, \quad \dot{y} = 2x + y,$

(b) $\dot{x} = 4x + 4y, \quad \dot{y} = x + 3y,$

(c) $\dot{x} = -x + 3y, \quad \dot{y} = -2x - 5y,$

(d) $\dot{x} = 2x - y + 4, \quad \dot{y} = 9x - 4y - 3,$

(e) $\dot{x} = 2x - 3y + 4, \quad \dot{y} = 2x - 2y + 3.$

Detailed sketches of the phase curves are not required.

Question 2 – 10 marks

Write the following second-order differential equation

$$\ddot{x} + \dot{x} + \alpha^2(x^2 - 1) = 0,$$

where α is a real, positive constant, as a pair of coupled first-order equations. Find its fixed points and determine how the classification of these fixed points changes with α.

Question 3 – 16 marks

Describe the qualitative features of the motion of a particle of unit mass moving in the following potentials. In each case sketch the potential, the phase diagram and give the equations of any separatrices that exist.

(a) $V(x) = (x^2 - 1)^2,$

(b) $V(x) = x^2/(1 + x^3), \quad x > -1.$

Question 4 – 25 marks

(a) Show that the system

$$\dot{x} = \alpha(y)f(x), \quad \dot{y} = \beta(x)f(y),$$

where α, β and f are sufficiently well-behaved but arbitrary functions, is generally not a Hamiltonian system in the (x, y)-representation.

Show that in the special case

$$\frac{df}{dz} = c\alpha(z) \quad \text{and} \quad \beta(z) = -\alpha(z),$$

where c is an arbitrary constant, the system *is* Hamiltonian.

(b) By making a change of dependent variables, $x = F(q)$ and $y = G(p)$, show that it is possible to find functions G and F such that in the (q, p)-representation the system is Hamiltonian.