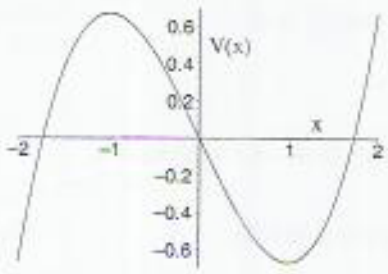
M821: TMA01 2001
Solution for question 2

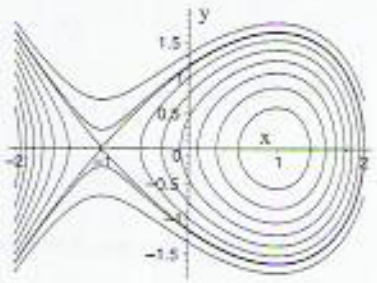
First observe that this equation is related to the conservative system

$$\ddot{x} + \alpha^2(x^2 - 1) = 0$$

with potential $V(x) = \alpha^2 x(x^2/3 - 1)$, having the shape shown in the left hand figure, on the right some representative phase curves are shown.



The potential $V(x)$ for $\alpha = 1$.



Phase curves for $\alpha = 1$.

This conservative system has a saddle at $(-1, 0)$ and a centre at $(1, 0)$. Its phase curves are the contours of the energy integral, JS section 1.3 page 10,

$$E = \frac{1}{2}y^2 + V(x), \quad y = \dot{x}.$$

From the shape of the potential we see that motion is possible for all real values of E but for $\min(V) \leq E \leq \max(E)$, that is $-2\alpha^2/3 \leq E \leq 2\alpha^2/3$, bound, periodic motion is possible. Some typical phase curves are shown in the right hand figure.

Now consider the original system by putting $y = \dot{x}$:

$$\dot{x} = y, \quad \dot{y} = -y + \alpha^2(1 - x^2)$$

which also has fixed points at $(-1, 0)$ and $(1, 0)$. The linearisation matrix is

$$A = \begin{pmatrix} 0 & 1 \\ -2\alpha^2 x & -1 \end{pmatrix}.$$ [2 marks]

Thus at $(-1, 0)$ the eigenvalues of A are

$$\lambda = \frac{1}{2}\left(-1 \pm \sqrt{1 + 8\alpha^2}\right)$$

which are real and of opposite sign. This fixed point is therefore a saddle. That is the introduction of the damping term, $\dot{x}$, has not chanmged the nature of this fixed point. [2 marks]