M821, TMA01 2001
Solution for question 4

(i) The system is Hamiltonian if there exists a *Hamiltonian function*, $H(x, y)$, such that

$$\begin{aligned} \dot{x} &= \alpha(y)f(x) = \frac{\partial H}{\partial y}, \\ \dot{y} &= \beta(x)f(y) = -\frac{\partial H}{\partial x}. \end{aligned}$$

From these two equations we can form two expressions for $\partial^2 H/\partial x \partial y$ which must be identical. From the first equation we have

$$\frac{\partial^2 H}{\partial x \partial y} = \alpha(y)f'(x) \quad \text{and from the second} \quad \frac{\partial^2 H}{\partial y \partial x} = -\beta(x)f'(y).$$

Thus we must have

$$\frac{f'(y)}{\alpha(y)} = -\frac{f'(x)}{\beta(x)}. \quad (1)$$

$\Rightarrow f'(y) = c\alpha(y)$

The first ratio is a function of y only and the second ratio is a function of x, so the only way these can be equal for all x and y is if they both equal the same constant, that is

$$\frac{df}{dz} = c\alpha(z), \quad \beta(z) = -\alpha(z)$$

$\Rightarrow \frac{\partial H}{\partial y} = \frac{1}{c} f'(y) f(x)$

for some constant c.

These relations show that the functions f, α and β need to be related for the system to be Hamiltonian; for arbitrary functions equation (1) will not hold.

If equation (1) is satisfied then

$-f'(x) = \beta(x)c$

$$\frac{\partial H}{\partial y} = \frac{1}{c}f(x)f'(y) \quad \text{and} \quad \frac{\partial H}{\partial x} = \frac{1}{c}f(y)f'(x) \quad \text{so} \quad H(x, y) = \frac{1}{c}f(x)f(y). \quad (2)$$

$\Rightarrow \partial H/\partial x = \frac{1}{c} f'(x) f(y)$

(ii) With the change of variables given the equations of motion become

$$\dot{q} = \alpha(G)f(F)/F'(q), \quad \dot{p} = \beta(F)f(G)/G'(p).$$

* $\dot{q}$ independent of $q \Rightarrow \dot{q} = k\alpha(G)$. Take $k = 1$ then $1 = f(F) F'(q)$.

The trick now is to note that if $\dot{q}$ is independent of q and $\dot{p}$ is independent of p then there exists a Hamiltonian of the form $H(q, p) = h_1(q) + h_2(p)$. So we define $F(q)$ and $G(p)$ by the relations,

$$\frac{dF}{dq} = f(F) \quad \text{or} \quad q = \int \frac{dF}{f(F)},$$

and

$$\frac{dG}{dp} = f(G) \quad \text{or} \quad p = \int \frac{dG}{f(G)}.$$

$\dot{p}$ independent of $p \Rightarrow \dot{p} = \beta(F)$ and $1 = f(G) G'(p)$