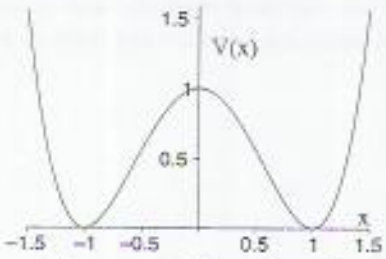
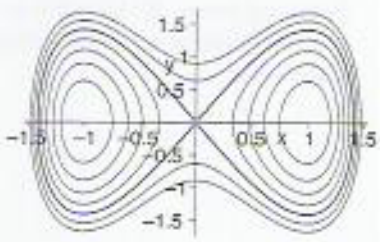
M821 TMA01 2001
Solution for question 3

Part (i)

The potential $V(x) = (x^2 - 1)^2$ is zero at $x = \pm 1$ and positive elsewhere. Thus at $x = \pm 1$ it has minima and in between must have a maximum which, by symmetry, must be at $x = 0$ and here $V(0) = 1$. For $|x| \gg 1$, $V(x) \sim x^4$. The graph of the potential is shown in the left hand figure.



The potential $V = (x^2 - 1)^2$.



Phase curves for the potential $V = (x^2 - 1)^2$.

[1 for potential
[3 for phase curve

The energy equation is

$$E = \frac{1}{2}y^2 + V(x), \quad y = \frac{dx}{dt},$$

so motion is possible only if $E \geq 0$. The graph of the potential shows that there are centres at $x = \pm 1$ and a saddle at $x = 0$.

If $E > V(0) = 1$ the phase curves enclose the origin and the motion is periodic. If $0 \leq E < 1$ motion is restricted to either the left or the right of the origin; both types of motion are periodic. Some typical phase curves are shown in the right hand figure. [2 marks for discussion]

The phase curve with energy $E = V(0) = 1$ is special, being the separatrix that divides the phase space into these three types of motion. For this potential the only non-periodic motion is represented by the separatrix; all other motion is periodic. The equation of the separatrix, shown by the thick line in the above figure, is

$$\frac{1}{2}y^2 = x^2(2 - x^2), \quad \text{or} \quad y = \pm x\sqrt{2(2 - x^2)}.$$

[1 mark for separatrix]

[Total 7]

Part (ii)

The potential $V(x) = x^2/(1 + x^3)$ is infinite at $x = -1$. Near the origin it behaves as $V(x) = x^2 + O(x^5)$ and $V(x) \to 0$ as $x \to \infty$, so it has a maximum for some positive value of x. The graph of this potential is shown in the following figure.