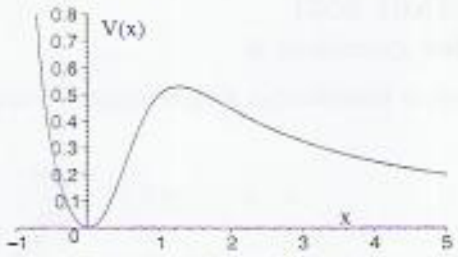


The potential $V = x^2/(1+x^3)$.

[1 for potential]

The maximum of this potential is at the root of

$$V'(x) = \frac{x(x^3-2)}{(1+x^3)^2} \quad \text{that is} \quad x = 2^{1/3} \quad \text{and} \quad \max(V) = \frac{2^{2/3}}{3} \simeq 0.529$$

This graph of the potential shows that the system has a centre at $x = 0$ and a saddle at $x = 2^{1/3}$. [2 marks]

The energy equation is

$$E = \frac{1}{2}y^2 + V(x), \quad y = \frac{dx}{dt},$$

so motion is possible only if $E \geq 0$, and the graph of the potential shows that there are three possible types of motion.

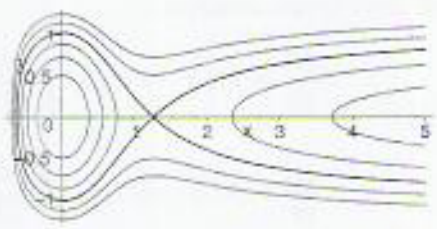
1) If $0 \leq E < \max(V)$ there is both bound, periodic motion and unbounded motion. The bound motion oscillates about the potential minimum at $x = 0$ and the unbounded motion is reflected off the potential barrier, not having sufficient energy to pass over the potential maximum. [1 mark]

2) If $E > \max(V)$ there is only unbounded motion which bounces off the barrier due to the singularity in the potential at $x = -1$. [1 mark]

The phase curve with energy $E = \max(V)$ is special, being the separatrix that divides phase space into the regions supporting these three types of motion. The equation of the separatrix is,

$$y = \pm 2^{5/6}\left(x - 2^{1/3}\right)\sqrt{\frac{x + 2^{-2/3}}{x^3+1}}.$$ [1 mark]

Some representative phase curves are shown in the figure.



Phase curves for the potential $V(x) = x^2/(1+x^3)$.

[3 for phase curves]

[Total 9]