M821, TMA01 2001
Solution for question 1

All these systems are linear and can be written in the form

$$\dot{\mathbf{x}} = A\mathbf{x},$$

possibly after a translation to put the fixed point at the origin, where A is a 2×2 matrix. The nature of the fixed point depends only upon $\det(A)$ and $\mathrm{Tr}(A)$ via the eigenvalues of the equation

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0.$$

a) In this case

$$\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{giving} \quad \det(A) = -1, \quad \mathrm{Tr}(A) = 4.$$

The eigenvalues are $\lambda = 2 \pm \sqrt{5}$, giving a saddle. [2 marks]

b) In this case

$$\begin{pmatrix} 4 & 4 \\ 1 & 3 \end{pmatrix} \quad \text{giving} \quad \det(A) = 8, \quad \mathrm{Tr}(A) = 7.$$

The eigenvalues are $\lambda = (7 \pm \sqrt{17})/2$, giving an unstable node. [2 marks]

c) In this case

$$\begin{pmatrix} -1 & 3 \\ -2 & -5 \end{pmatrix} \quad \text{giving} \quad \det(A) = 11, \quad \mathrm{Tr}(A) = -6.$$

The eigenvalues are $\lambda = -3 \pm i\sqrt{2}$, giving a stable spiral.* [2 marks]

d) In this case

$$\begin{pmatrix} 2 & -1 \\ 9 & -4 \end{pmatrix} \quad \text{giving} \quad \det(A) = 1, \quad \mathrm{Tr}(A) = -2.$$

The eigenvalues are identical, $\lambda = -1$, giving a stable improper node. [2 marks]

e) In this case

$$\begin{pmatrix} 2 & -3 \\ 2 & -2 \end{pmatrix} \quad \text{giving} \quad \det(A) = 2, \quad \mathrm{Tr}(A) = 0.$$

The eigenvalues are $\lambda = \pm i\sqrt{2}$, giving a centre. [2 marks]

* Please note that this line in the previous version of this solution was incorrect.